Kinematic dynamo in a random flow with strong average shear

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We analyze kinematic dynamo effect in a conducting fluid where stationary shear flow is accompanied by relatively weak random smooth velocity fluctuations. Then ....

I. INTRODUCTION

Dynamo implies magnetic field generation in a conducting fluid where a random flow is present. We consider the case where the magnetic field is growing from weak fluctuations and examine an initial stage of the magnetic field evolution when its amplitude is small, so that one can neglect feedback from the magnetic field to the flow. This stage is called kinematic one, and the magnetic field is passive at the stage. We assume that the random flow is statistically homogeneous in space and time, and analyze an evolution of magnetic fluctuations in this case.

We consider an evolution of a random in space magnetic field with characteristic scale much less than the velocity correlation length. Then the magnetic field is increasing at the kinematic stage, the increase can be characterized by single-point moments of the magnetic field induction $\boldsymbol{B}$ that grow exponentially in time:

$$\langle \boldsymbol{B}^{2n}(t)\rangle \propto \exp\left(\gamma_n t\right). \tag{1}$$

Here angular brackets mean averaging over space and γ_n are increments, that are subjects of our investigation.

The increments γ_n are determined by the random flow, they are related to the Lyapunov exponent λ that is average logarithmic diverging rate of close fluid particles. The quantity λ^{-1} plays the role of the Lagrangian correlation time of the random flow. Therefore we are interested in times $t \gg \lambda^{-1}$ when the laws (1) are well observed.

The flow is assumed to be composed of a steady shear flow and a random component. Examples!
$d = 3$ versus $d = 2$ and other dimensions.
Elastic turbulence
Magneto-diffusion
$\nabla \boldsymbol{v} = 0$

II. BASIC RELATIONS

The magnetic field evolution in a conducting medium is governed by the equation

$$\partial_t \boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \kappa\nabla^2\boldsymbol{B}. \tag{2}$$

Here $\boldsymbol{v}$ is the flow velocity and κ is the magneto-diffusion coefficient, inversely proportional to the electrical conductivity of the medium. We consider the case where the magnetic field is weak and, therefore, κ is a constant independent of the magnetic field. The flow is assumed to be incompressible, $\nabla \boldsymbol{v} = 0$.

We investigate the magnetic field in a random (turbulent) flow assuming that the velocity statistics is homogeneous in space and time. The magnetic fluctuations are examined on scales much less than the velocity correlation length. Then the velocity field can be treated as smooth that is the velocity can be represented as a Taylor series. First terms of the expansion in a vicinity of a fluid particle are

$$v_j = V_j(t) + \Sigma_{ji}(t) r_i, \tag{3}$$

where $\boldsymbol{V}(t)$ is the fluid particle velocity and $\boldsymbol{r}$ is the radius-vector in the reference system with the origin in the fluid particle. Here Σ_{ji} is the velocity gradient matrix, due to the incompressibility condition its trace is equal to zero.

The flow is assumed to be composed of a steady shear flow and a random component. Then the matrix of the velocity gradients Σ_{ji} is written as

$$\Sigma_{ji}(t) = s\delta_{j1}\delta_{i2} + \sigma_{ji}(t), \tag{4}$$

where the axis one is directed along the shear velocity that varies along the axis two, and s is the shear coefficient. The random component σ_{ji} is zero in average and should be, consequently, characterized in terms of its pair correlation function, $\langle \sigma_{ji}(t_1)\sigma_{mn}(t_2)\rangle$ that is a function of $t = t_1 - t_2$ due to the statistics homogeneity in time. We assume that the steady shear constituent of the flow is stronger than the random one, that is

$$s \gg \int dt\ \langle \sigma_{ij}(t)\sigma_{mn}(0)\rangle, \tag{5}$$

for all components of the matrix $\hat{\sigma}$.

Let us pass to the reference system attached to our marked fluid particle and make Fourier transform for the magnetic field. Then the equation (2) leads to

$$\partial_t B_i(t,\boldsymbol{k}) = \Sigma_{ij} B_j(t,\boldsymbol{k}) + \Sigma_{nj} k_n \frac{\partial}{\partial k_j} B_i(t,\boldsymbol{k}) - \kappa k^2 B_i(t,\boldsymbol{k}). \tag{6}$$

The equation can be explicitly solved in terms of the evolution matrix $\hat{W}$, that is the following chronologically ordered exponent

$$\hat{W}(t) = \mathrm{T}\exp\left[\int_0^t dt'\, \hat{\Sigma}(t')\right]. \tag{7}$$

The solution is written as

$$\boldsymbol{B}(t,\boldsymbol{k}) = \hat{W}(t)\mathcal{B}\left(\hat{W}^T\boldsymbol{k}\right) \times \tag{8}$$
$$\times \exp\left[-\kappa \int_0^t dt'\, \boldsymbol{k}\cdot \hat{W}(t)\{\hat{W}(t')\}^{-1}\{\hat{W}^T(t')\}^{-1}\hat{W}^T(t)\boldsymbol{k}\right],$$

where $\mathcal{B}(\boldsymbol{k})$ is Fourier-transform of the initial magnetic induction, existing at $t = 0$, and the subscript T labels transposed matrices.

III. PROPERTIES OF THE EVOLUTION MATRIX

Before passing to the moments of the magnetic field induction we overlook properties of the evolution matrix (7). First of all, determinant of $\hat{W}$ is equal to unity since the matrix $\hat{\Sigma}$ is traceless (due to incompressibility). Next, eigen values of $\hat{W}$ are obviously positive.

We are interested in properties of the evolution matrix $\hat{W}$ on times t larger than the inverse Lyapunov exponent, where $\hat{W}$ possesses some universal properties. They can be conveniently expressed in terms of the Gaussian decomposition

$$\hat{W}(t) = \hat{T}_L \hat{\Delta} \hat{T}_R, \tag{9}$$

where $\hat{T}_L$ and $\hat{T}_R$ are triangle matrices and $\hat{\Delta}$ is diagonal matrix:

$$\hat{T}_L = \begin{pmatrix} 1 & \chi & \chi_1 \\ 0 & 1 & \chi_2 \\ 0 & 0 & 1 \end{pmatrix}, \hat{T}_R = \begin{pmatrix} 1 & 0 & 0 \\ \zeta_1 & 1 & 0 \\ \zeta_2 & \zeta_3 & 1 \end{pmatrix}, \tag{10}$$

$$\hat{\Delta} = \begin{pmatrix} e^{-\rho_1} & 0 & 0 \\ 0 & e^{\rho} & 0 \\ 0 & 0 & e^{\rho_1 - \rho} \end{pmatrix}. \tag{11}$$

The structure of the diagonal matrix $\hat{\Delta}$ is determined by the condition $\det\hat{\Delta} = 1$, following from $\det\hat{W} = 1$.

Substituting the decomposition (9) into the equation $\partial_t \hat{W} = \hat{\Sigma}\hat{W}$, following from Eq. (7), one obtains

$$\hat{T}_L^{-1}\hat{\Sigma}\hat{T}_L = \hat{T}_L^{-1}\partial_t \hat{T}_L + \partial_t \hat{\Delta}\hat{\Delta}^{-1} + \hat{\Delta}\partial_t \hat{T}_R \hat{T}_R^{-1}\hat{\Delta}^{-1}. \tag{12}$$

Based on the relation one concludes that at $t \gg \lambda^{-1}$ the elements of the matrix $\hat{\Delta}$ behave exponentially in time, the $\hat{T}_R$ is frozen (being remained of order unity), whereas elements of the matrix $\hat{T}_L$ fluctuate, possessing a homogeneous in time statistics. The statistics of the parameters ρ and ρ_1 is determined by the central limit theorem since they are integrals of the random functions with homogeneous in time statistics.

The time derivative $\partial_t \rho$ can be estimated as λ. If $\lambda \ll s$ then a hierarchy $\chi \gg \chi_1 \gg \chi_2$ is correct as one can check from Eq. (12). Therefore in the main approximation in λ/s the only component $\sigma_{21} \equiv \sigma$ is relevant and the system of equations (12) is reduced to

$$\dot{\chi} = s - \chi^2 \sigma, \quad \dot{\rho} = \chi\sigma, \quad \rho_1 = -\rho. \tag{13}$$

The short correlated case is examined in Appendix A.

IV. RANDOM INITIAL CONDITIONS

We consider the case where the initial magnetic field is distributed randomly in space with homogenous in space statistics and have a single correlation length l. Then the induction pair correlation function can be written as

$$\langle \mathcal{B}_i(\boldsymbol{k})\mathcal{B}_j(\boldsymbol{k}')\rangle = \delta(\boldsymbol{k}+\boldsymbol{k}')\left(\delta_{ij}k^2 - k_i k_j\right) f(l^2 k^2), \tag{14}$$

where f is a function decaying fast as its argument tends to infinity. In our assumptions, a characteristic wave number of $\mathcal{B}$ should be much larger than the inverse velocity correlation length. Therefore l should be much smaller than the velocity correlation length.

При больших t характерные $\rho \gg 1$. В этом пределе из выражения (8) легко получается оценка:

$$\boldsymbol{B}^2(t) \sim \frac{l}{r_d}\boldsymbol{B}_0^2 e^{\rho}, \tag{15}$$

где r_d– диффузионная длина. Будем вычислять моменты $\langle \boldsymbol{B}^{2n}(t)\rangle$ для $n \gg 1$. В этом случае ведущий вклад в инкремент γ_n, see Eq. (1), определяется перевальной траекторией $\sigma(t) = const$, так что из уравнений (13) следует:

$$\rho = t\sqrt{s\sigma}.$$

Для определения γ_n нужно знать вероятность $p(\sigma)$ такой реализации $\sigma(t) = const$. Для времен t, много больших времени корреляции случайной переменной $\sigma(t)$, применимо обобщение закона больших чисел (следующее, фактически, из аддитивности энтропии):

$$p(\sigma) \sim \exp\left(-t\Gamma(\sigma)\right), \tag{16}$$

где $\Gamma(\sigma)$ – так называемая функция Крамера. Усреднение по σ выполняется также перевальным образом, и результат может быть представлен как

$$\gamma_n = n\sqrt{s\sigma_n} - \Gamma(\sigma_n), \tag{17}$$

где σ_n является решением перевального уравнения:

$$\Gamma'(\sigma_n) = \frac{n}{2}\sqrt{\frac{s}{\sigma_n}}. \tag{18}$$

Если время корреляции случайного процесса $\sigma(t)$ достаточно мало (заметно меньше, чем λ^{-1}), то функция Крамера является просто параболой:

$$\Gamma(\sigma) = \sigma^2/D, \tag{19}$$

и мы приходим к явному выражению для асимптотических при $n \gg 1$ значений искомых инкрементов:

$$\gamma_n = C\lambda n^{4/3}, \qquad C = 3\times 4^{4/3}. \tag{20}$$

V. CONCLUSION

So, ...

Приложение A

Here we consider the random constituent of the flow that is short correlated in time. As we noted, at the condition $\lambda \ll s$ the only relevant component of the matrix of the random velocity gradients is $\sigma \equiv \sigma_{21}$. In the short correlated case it can be treated as white noise that is

$$\langle \sigma(t_1)\sigma(t_2)\rangle = D\delta(t_1 - t_2), \tag{A1}$$

where the factor D characterizes strength of the noise. The condition $\lambda \ll s$ is equivalent to $D \ll s$. Introducing the variable $x = \chi^{-1}$, one obtains from Eqs. (13,A1) the following Fokker-Plank equation for the probability distribution function $P(x)$:

$$\partial_t P = \partial_x(sx^2P) + (D/2)\partial_x^2 P. \tag{A2}$$

A stationary solution of the equation is

$$P \propto \exp\left(-\frac{2sx^2}{3D}\right), \tag{A3}$$

that leads to

$$\langle x \rangle = \langle \chi^{-1} \rangle = \frac{\Gamma(2/3)}{\Gamma(1/3)} \left(\frac{3D}{2s} \right)^{1/3} . \tag{A4}$$

And then we obtain from Eq. (13)

$$\lambda = \langle \chi \sigma \rangle = \langle s x \rangle = \frac{\Gamma(2/3)}{\Gamma(1/3)} \left(\frac{3Ds^2}{2} \right)^{1/3} . \tag{A5}$$

[1] Л.Д.Ландау, Е.М.Лифшиц, Электродинамика сплошных сред.