Multidimensional integrable hierarchies connected with vector fields

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Outline

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Introduction

- Linear operators (Lax pairs) – vector fields, meromorphyc in spectral variable and including derivative ∂_λ over spectral variable (N 'space' variables + spectral variable)
- Connected to nonlinear vector Riemann-Hilbert problem of the type

$$\mathbf{\Psi}_+ = \mathbf{F}(\mathbf{\Psi}_-),$$

  where $\mathbf{\Psi}_+$, $\mathbf{\Psi}_-$ are boundary values of (N+1)-component vector function on the sides of some curve (e.g. unit circle)
- One singular point case – dispersionless KP hierarchy, second heavenly equation hierarchy, Dunajski system hierarchy. Special vector fields (area-preserving or volume preserving). A simplest general position case (N=1) is connected with Manakov-Santini system.
- Two singular points case for N=1 and Hamiltonian vector fields correspond to dispersionless 2DTL hierarchy. We consider this case for general vector fields.

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- + Extensive literature on dispersionless integrable hierarchies

Dressing scheme

vector Riemann problem of the form

$$\mathbf{\Psi}^{+} = \mathbf{F}(\mathbf{\Psi}^{-}),$$

where $\mathbf{\Psi}^{+}$, $\mathbf{\Psi}^{-}$ denote boundary values of the (N+1)-component vector function (column) on the sides of some oriented curve γ in the complex plane of the variable λ.

Linearized problem

$$\delta\mathbf{\Psi}^{+} = \frac{D(\mathbf{F})}{D(\mathbf{\Psi})}\delta(\mathbf{\Psi}^{-}), \quad \frac{D(\mathbf{F})}{D(\mathbf{\Psi})}_{ij} = \left(\frac{\partial F_i}{\partial \Psi_j}\right)$$

Let $\mathbf{\Psi}$ depend on a set of extra variables t_n (times). Then $\frac{\partial}{\partial t_n}\mathbf{\Psi}$, and also $\lambda^k \frac{\partial}{\partial t_n}\mathbf{\Psi}$, $\lambda^k \frac{\partial}{\partial \lambda}\mathbf{\Psi}$, satisfy linearized problem, and, suggesting uniqueness of its solution (or absence of nonzero analytic solutions), it is possible to develop a scheme of constructing linear equations, which give nonlinear integrable PDEs as compatibility conditions.

Generating relation

Let us consider a differential form

$$\omega = \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \dots \wedge \mathrm{d}\Psi^N,$$

where the differential includes both times and a spectral variable, $\mathrm{d}f = \sum_{n=1}^{\infty} \partial_n f \mathrm{d}t_n + \partial_\lambda f \mathrm{d}\lambda$. The condition on the curve γ is

$$\omega^+ = \left| \frac{D(\mathbf{F})}{D(\mathbf{\Psi})} \right| \omega^-$$

Let us fix a set of N+1 variables $\lambda, t_1, \dots, t_n$ and introduce a Jacobi matrix

$$J_{ij} = \partial_i \Psi^j, \quad 0 \leqslant i, j \leqslant N, \quad \partial_0 = \frac{\partial}{\partial \lambda}, \ \partial_k = \frac{\partial}{\partial t_k}, \quad 1 \leqslant k \leqslant N.$$

Then for normalized form $\Omega = (\det J)^{-1} \omega$

$$\Omega^+ = \Omega^-$$

Generating relation, plays the role similar to Hirota bilinear identity.

General (N+2)-dimensional one-point hierarchy

N+1 formal series depending on N infinite sets of 'times'

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \dots, \mathbf{t}^N)\lambda^{-n},$$

$$\Psi^k = \sum_{n=0}^{\infty} t^k_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^k_n(\mathbf{t}^1, \dots, \mathbf{t}^N)(\Psi^0)^{-n}.$$

where $1 \leqslant k \leqslant N$, $\mathbf{t}^k = (t^k_0, \dots, t^k_n, \dots)$. We denote $\partial^k_n = \frac{\partial}{\partial t^k_n}$, $\mathbf{\Psi}$ – vector (column) with components $\Psi^0, \dots, \Psi^N$, projectors $(\sum_{-\infty}^{\infty} u_n \lambda^n)_+ = \sum_{n=0}^{\infty} u_n \lambda^n$, $(\sum_{-\infty}^{\infty} u_n \lambda^n)_- = \sum_{-\infty}^{n=-1} u_n \lambda^n$.

Generating relation

The hierarchy is defined by the generating relation

$$(J_0^{-1}\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \cdots \wedge \mathrm{d}\Psi^N)_- = 0, \qquad (1)$$

where the differential includes times and spectral variable,

$$\mathrm{d}f = \sum_{k=1}^{N}\sum_{n=0}^{\infty} \partial_n^k f \mathrm{d}t_n^k + \partial_\lambda f \mathrm{d}\lambda,$$

and J_0 – determinant of Jacobi matrix J,

$$J_{ij} = \partial_i \Psi^j, \quad 0 \leqslant i, j \leqslant N, \quad \partial_0 = \frac{\partial}{\partial\lambda},\ \partial_k = \frac{\partial}{\partial x^k}, \quad 1 \leqslant k \leqslant N,$$

where $x^k = t_0^k$. Relation (1) generates Lax-Sato equations.

Proposition

Relation (1) is equivalent to the infinite set of Lax-Sato equations

$$\partial_n^k \mathbf{\Psi} = \sum_{i=0}^{N} (J_{ki}^{-1} (\Psi^0)^n)_+ \partial_i \mathbf{\Psi}, \quad 0 \leqslant n \leqslant \infty \,, 1 \leqslant k \leqslant N. \qquad (2)$$

(1) $\Rightarrow$ hierarchy (2) follows from

Lemma

Given generating relation (1)$\Rightarrow$ for arbitrary first order operator $\hat{U}$,

$$\hat{U} = \sum_{k=1}^{N} \sum_{i} u_i^k(\lambda, \mathbf{t}) \partial_i^k + u^0(\lambda, \mathbf{t}^1, \mathbf{t}^2) \partial_\lambda$$

with 'plus' coefficients ($(u_i^k)_- = u_-^0 = 0$), the condition $(\hat{U}\mathbf{\Psi})_+ = 0$ (where for Ψ^k, $k \neq 0$ derivatives are taken for fixed Ψ^0) implies that $\hat{U}\Psi = 0$.

Simplest equations

The basis $\lambda^n\partial_k\Psi$, $\lambda^n\partial_\lambda\Psi$, $0 \leqslant n < \infty$, $0 < k \leqslant N$. We expand $\partial_1^k\Psi$ into the basis

$$\partial_1^k\Psi = (\lambda\partial_k - \sum_{p=1}^{N}(\partial_k u_p)\partial_p - (\partial_k u_0)\partial_\lambda)\Psi, \quad 0 < k \leqslant N,$$

Compatibility condition for the pair of flows with ∂_1^k and ∂_1^q, $k \neq q$

$$\partial_1^k\partial_q\hat{u} - \partial_1^q\partial_k\hat{u} + [\partial_k\hat{u}, \partial_q\hat{u}] = (\partial_k u_0)\partial_q - (\partial_q u_0)\partial_k,$$
$$\partial_1^k\partial_q u_0 - \partial_1^q\partial_k u_0 + (\partial_k\hat{u})\partial_q u_0 - (\partial_q\hat{u})\partial_k u_0 = 0,$$

where $\hat{u}$ is a vector field, $\hat{u} = \sum_{p=1}^N u_k\partial_k$. The case N=2 + volume preservation reduction gives Dunajski system.
For $u_0 = 0$ we have only the first equation without the rhs. This case corresponds to general vector fields without the derivative on spectral variable, and, after Hamiltonian reduction, represents hyper-Kahler hierarchies (Takasaki).

Dunajski equation

A canonical Plebański form of null-Kähler metrics (signature (2,2))

$$g = \mathrm{d}w\mathrm{d}x + \mathrm{d}z\mathrm{d}y - \Theta_{xx}\mathrm{d}z^2 - \Theta_{yy}\mathrm{d}w^2 + 2\Theta_{xy}\mathrm{d}w\mathrm{d}z. \tag{3}$$

The conformal anti-self-duality (ASD) condition leads to Dunajski equation

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f, \tag{4}$$

$$\Box f = f_{xw} + f_{yz} + \Theta_{yy} f_{xx} + \Theta_{xx} f_{yy} - 2\Theta_{xy} f_{xy} = 0. \tag{5}$$

Linear system $L_0\Psi = L_1\Psi = 0$, where $\Psi = \Psi(w, z, x, y, \lambda)$ and

$$\begin{aligned} L_0 &= (\partial_w - \Theta_{xy}\partial_y + \Theta_{yy}\partial_x) - \lambda\partial_y + f_y\partial_\lambda, \\ L_1 &= (\partial_z + \Theta_{xx}\partial_y - \Theta_{xy}\partial_x) + \lambda\partial_x - f_x\partial_\lambda. \end{aligned}$$

The case $f = 0$ corresponds to metrics of the form (3) satisfying Einstein equations, and Dunajski equation (4), (5) reduces to Plebański second heavenly equation.

Dunajski equation hierarchy

General one-point hierarchy for N=2 + reduction $J_0 = 1$

$$\Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},$$

$$\Psi^1 = \sum_{n=0}^{\infty} t^1_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^1_n(\mathbf{t}^1, \mathbf{t}^2)(\Psi^0)^{-n}$$

$$\Psi^2 = \sum_{n=0}^{\infty} t^2_n (\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^2_n(\mathbf{t}^1, \mathbf{t}^2)(\Psi^0)^{-n}.$$

Generating relation

$$(\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2)_- = 0$$

Lax-Sato equations of Dunajski hierarchy

$$\partial_n^1\Psi = +\left((\Psi^0)^n \begin{vmatrix}\Psi^0_\lambda & \Psi^2_\lambda\\ \Psi^0_y & \Psi^2_y\end{vmatrix}\right)_+ \partial_x\Psi - \left((\Psi^0)^n \begin{vmatrix}\Psi^0_\lambda & \Psi^2_\lambda\\ \Psi^0_x & \Psi^2_x\end{vmatrix}\right)_+ \partial_y\Psi - \left((\Psi^0)^n \begin{vmatrix}\Psi^0_x & \Psi^2_x\\ \Psi^0_y & \Psi^2_y\end{vmatrix}\right)_+ \partial_\lambda\Psi,$$

$$\partial_n^2\Psi = -\left((\Psi^0)^n \begin{vmatrix}\Psi^0_\lambda & \Psi^1_\lambda\\ \Psi^0_y & \Psi^1_y\end{vmatrix}\right)_+ \partial_x\Psi + \left((\Psi^0)^n \begin{vmatrix}\Psi^0_\lambda & \Psi^1_\lambda\\ \Psi^0_x & \Psi^1_x\end{vmatrix}\right)_+ \partial_y\Psi + \left((\Psi^0)^n \begin{vmatrix}\Psi^0_x & \Psi^1_x\\ \Psi^0_y & \Psi^1_y\end{vmatrix}\right)_+ \partial_\lambda\Psi$$

(plus a condition $J_0 = 1$). For $\Psi^0 = \lambda$ Dunajski equation hierarchy reduces to second heavenly equation hierarchy (Takasaki), while for $\Psi^2 = y$ it reduces to dispersionless KP hierarchy

First two flows

$$\partial_1^1\Psi = (u_y\partial_x - u_x\partial_y + \lambda\partial_x - f_x\partial_\lambda)\Psi,$$
$$\partial_1^2\Psi = (v_x\partial_y - v_y\partial_x + \lambda\partial_y - f_y\partial_\lambda)\Psi,$$

where

$$u = \Psi_1^2, \quad v = \Psi_1^1, \quad f = \Psi_1^0.$$

$\det J_0 = 1 \Rightarrow u_y + v_x = 0$, then we introduce a potential Θ, $v = \Theta_y$, $u = -\Theta_x$. After the identification $z = -t_1^1$, $w = t_1^2$ we get Lax pair for Dunajski equation

$$\Theta_{wx} + \Theta_{zy} + \Theta_{xx}\Theta_{yy} - \Theta_{xy}^2 = f,$$
$$f_{xw} + f_{yz} + \Theta_{yy}f_{xx} + \Theta_{xx}f_{yy} - 2\Theta_{xy}f_{xy} = 0.$$

Manakov-Santini system

$$
\begin{aligned}
u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\
v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y,
\end{aligned} \tag{6}
$$

Lax pair

$$
\begin{aligned}
&\partial_y \Psi = ((\lambda - v_x)\partial_x - u_x \partial_\lambda)\Psi, \\
&\partial_t \Psi = ((\lambda^2 - v_x\lambda + u - v_y)\partial_x - (u_x\lambda + u_y)\partial_\lambda)\Psi.
\end{aligned}
$$

For $v = 0$ reduces to dKP (Khohlov-Zabolotskaya equation)

$$
u_{xt} = u_{yy} + (uu_x)_x, \tag{7}
$$

reduction $u = 0$ gives the equation (Pavlov)

$$
v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y. \tag{8}
$$

Manakov-Santini system hierarchy

General hierarchy with $N = 1$

$$L = \Psi^0 = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t})\lambda^{-n}, \tag{9}$$

$$M = \Psi^1 = \sum_{n=0}^{\infty} t_n(\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^1_n(\mathbf{t})(\Psi^0)^{-n}. \tag{10}$$

Generating relation

$$(J_0^{-1}\mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1)_- = 0, \tag{11}$$

where

$$J = \begin{pmatrix} \Psi^0_\lambda & \Psi^1_\lambda \\ \Psi^0_x & \Psi^1_x \end{pmatrix}, \quad J_0 = \det J = 1 + \partial_x\Psi^1_1\lambda^{-1} + (\partial_x\Psi^1_2 - \Psi^0_1)\lambda^{-2} + \dots .$$

Lax-Sato equations

$$\partial_n\mathbf{\Psi} = (J_0^{-1}\Psi^0_\lambda(\Psi^0)^n)_+\partial_x\mathbf{\Psi} - (J_0^{-1}\Psi^0_x(\Psi^0)^n)_+\partial_\lambda\mathbf{\Psi}.$$

Lax-Sato equations for the first two flows

$$\partial_y \mathbf{\Psi} = ((\lambda - v_x)\partial_x - u_x\partial_\lambda)\mathbf{\Psi},$$
$$\partial_t \mathbf{\Psi} = ((\lambda^2 - v_x\lambda + u - v_y)\partial_x - (u_x\lambda + u_y)\partial_\lambda)\mathbf{\Psi},$$

where $u = \Psi_1^0$, $v = \Psi_1^1$, $x = t_0$, $y = t_1$, $t = t_2$. Compatibility condition gives Manakov-Santini system (6).

To reduce Manakov-Santini hierarchy to dKP hierarchy, one should consider the condition $J_0 = 1$ (corresponds to Hamiltonian or area-preserving vector fields), then Lax-Sato equations of Manakov-Santini hierarchy directly reduce to Lax-Sato equations of dKP hierarchy. Respectively,the reduction $\Psi^0 = \lambda$ leads to the hierarchy connected with equation (8) (Pavlov), considered also by Martínez Alonso and Shabat.

Non-Hamiltonian 2DTL generalization

A simplest generalization of dispersionless 2DTL equation reads

$$\begin{aligned}(\mathrm{e}^{-\phi})_{tt} &= m_t\phi_{xy} - m_x\phi_{ty}, \\ m_{tt}\mathrm{e}^{-\phi} &= m_{ty}m_x - m_{xy}m_t,\end{aligned} \tag{12}$$

with a Lax pair

$$\partial_x\mathbf{\Psi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t\frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi},$$

$$\partial_y\mathbf{\Psi} = \left(\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t + \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}$$

For $m = t$ system (20) reduces to dispersionless 2DTL equation

$$(\mathrm{e}^{-\phi})_{tt} = \phi_{xy},$$

Respectively, $\phi = 0$ reduction gives an equation (Martínez Alonso and Shabat, Pavlov)

$$m_{tt} = m_{ty}m_x - m_{xy}m_t.$$

Generalized dispersionless 2DTL hierarchy

We generalize the scheme of dispersionless 2DTL hierarchy (Takasaki-Takebe).
Formal series ('+' may be associated with infinity, and '-' with zero, usually we suggest they define the functions outside and inside the unit circle),

$$\Lambda^+ = \ln\lambda + \sum_{k=1}^{\infty} l_k^+ \lambda^{-k}, \quad \Lambda^- = \ln\lambda + \phi + \sum_{k=1}^{\infty} l_k^- \lambda^k,$$

$$M^+ = M_0^+ + \sum_{k=1}^{\infty} m_k^+ \mathrm{e}^{-k\Lambda^+}, \quad M^- = M_0^- + m_0 + \sum_{k=1}^{\infty} m_k^- \mathrm{e}^{k\Lambda^-},$$

$$M_0 = t + x\mathrm{e}^{\Lambda} + y\mathrm{e}^{-\Lambda} + \sum_{k=1}^{\infty} x_k \mathrm{e}^{(k+1)\Lambda} + \sum_{k=1}^{\infty} y_k \mathrm{e}^{-(k+1)\Lambda}$$

Usually for simplicity we suggest that only finite number of x_k, y_k is not equal to zero.

Generating relation of the hierarchy

$$((J_0)^{-1}\mathrm{d}\Lambda \wedge \mathrm{d}M)^+ = ((J_0)^{-1}\mathrm{d}\Lambda \wedge \mathrm{d}M)^-, \tag{13}$$

where J_0 is a determinant of Jacobi type matrix J,

$$J = \begin{pmatrix} \lambda\partial_\lambda\Lambda & \partial_t\Lambda \\ \lambda\partial_\lambda M & \partial_t M \end{pmatrix},$$

the differential d takes into account all variables t, x, x_k, y_k and a spectral variable λ, and $(\dots)^+$, $(\dots)^-$ here are not projectors, but mean that all series have superscript '$+$' or '-' (or the functions are taken inside and outside the unit circle). As a result, the expression in generating relation is *meromorphic*.

Lax-Sato equations

$$\partial_n^+ \mathbf{\Psi} = \left((J_0^{-1}(\lambda\partial_\lambda\Lambda^+)\mathrm{e}^{(n+1)\Lambda^+})_+\partial_t - (J_0^{-1}(\partial_x\Lambda^+)\mathrm{e}^{(n+1)\Lambda^+})_+\partial_\lambda \right) \mathbf{\Psi},$$
$$\partial_n^- \mathbf{\Psi} = \left((J_0^{-1}(\lambda\partial_\lambda\Lambda^-)\mathrm{e}^{-(n+1)\Lambda^-})_-\partial_t - (J_0^{-1}(\partial_x\Lambda^+)\mathrm{e}^{-(n+1)\Lambda^+})_-\partial_\lambda \right) \mathbf{\Psi}$$

where

$$\mathbf{\Psi} = \begin{pmatrix} \Lambda \\ M \end{pmatrix}, \qquad \partial_n^+ = \frac{\partial}{\partial x_n}, \quad \partial_n^- = \frac{\partial}{\partial y_n},$$
$$(\sum_{k=-\infty}^{\infty} c_k\lambda^k)_- = \sum_{k=-\infty}^{-1} c_k\lambda^k, \qquad (\sum_{k=-\infty}^{\infty} c_k\lambda^k)_+ = \sum_{k=0}^{\infty} c_k\lambda^k.$$

The flows with ∂_0^+ and ∂_0^- give a Lax pair for generalized d2DTL , $m = m_0 + t$.
Condition $J_0 = 1$ reduces the hierarchy to d2DTL, condition $\Lambda = \ln\lambda$ to the hierarchy considered by Martínez Alonso and Shabat, also Pavlov.

Transformations. Symmetric generalization of elliptic d2DTL

We search for non-Hamiltonian generalization of ellyptic d2DTL

$$(\mathrm{e}^{-\phi})_{tt} = \phi_{z\bar{z}},$$

preserving the symmetry. Gauge transformation (present in d2DTL case, Takasaki), changes Lax pair, preserves equations

$$\lambda \to \lambda \exp(-\epsilon\phi),$$

where ϵ is a parameter. After this transformation we get Λ of the form

$$\Lambda^+ = \ln\lambda - \epsilon\phi + \sum_{k=1}^{\infty} l_k^+ \lambda^{-k},$$

$$\Lambda^- = \ln\lambda + (1-\epsilon)\phi + \sum_{k=1}^{\infty} l_k^- \lambda^{k}.$$

In the Lax pair one should perform a substitution

$$\lambda \to \lambda \exp(-\epsilon\phi),\ \partial_\lambda \to \exp(\epsilon\phi)\partial_\lambda.$$
$$\partial_x \to \partial_x + \epsilon\lambda\phi_x\partial_\lambda,\ \partial_y \to \partial_y + \epsilon\lambda\phi_y\partial_\lambda,\ \partial_t \to \partial_t + \epsilon\lambda\phi_t\partial_\lambda,$$

In elliptic d2DTL case for $\epsilon = \frac{1}{2}$ we get a symmetric Lax pair

$$\partial_z \boldsymbol{\Psi} = L_1 \boldsymbol{\Psi} = \left((\lambda \mathrm{e}^{-\frac{1}{2}\phi})\partial_t + \frac{1}{2}(\phi_z + \lambda \mathrm{e}^{-\frac{1}{2}\phi}\phi_t)\lambda\partial_\lambda \right) \boldsymbol{\Psi},$$
$$\partial_{\bar{z}} \boldsymbol{\Psi} = L_2 \boldsymbol{\Psi} = \left((\frac{1}{\lambda} \mathrm{e}^{-\frac{1}{2}\phi})\partial_t - \frac{1}{2}(\phi_{\bar{z}} + \lambda \mathrm{e}^{-\frac{1}{2}\phi}\phi_t)\lambda\partial_\lambda \right) \boldsymbol{\Psi}$$

On the unit circle $L_1 = \bar{L}_2$.

Reciprocal transformation

To get symmetric non-Hamiltonian generalization of ellyptic d2DTL and symmetric Lax pair for it, we should also use a reciprocal transformation

$$t = \tau - \alpha m_0$$

(where τ is a new 'time', α is a parameter), which gives M of the form

$$M^+ = M_0^+ + (1-\alpha)m_0 + \sum_{k=1}^{\infty} m_k^+ \mathrm{e}^{-k\Lambda^+},$$

$$M^- = M_0^- - \alpha m_0 + \sum_{k=1}^{\infty} m_k^- \mathrm{e}^{k\Lambda^+},$$

$$M_0 = \tau + x\mathrm{e}^{\Lambda} + y\mathrm{e}^{-\Lambda} + \dots$$

Derivatives transform as follows,

$$\partial_x \to \partial_x + \frac{\alpha m_{0x}}{1-\alpha m_{0\tau}}\partial_\tau,\ \partial_y \to \partial_y + \frac{\alpha m_{0y}}{1-\alpha m_{0\tau}}\partial_\tau,\ \partial_t \to \partial_\tau + \frac{\alpha m_{0\tau}}{1-\alpha m_{0\tau}}\partial_\tau.$$

Taking $x = z$, $y = \bar{z}$, $\epsilon = \frac{1}{2}$, $\phi \to -2\phi$, $\alpha = \frac{1}{2}$, $m_0 = -2\mathrm{i}m$, we get

$$\Lambda^+ = \ln\lambda + \phi + \sum_{k=1}^{\infty} l_k^+ \lambda^{-k}, \quad \Lambda^- = \ln\lambda - \phi + \sum_{k=1}^{\infty} l_k^- \lambda^{k},$$

$$M^+ = M_0^+ + \mathrm{i}m + \sum_{k=1}^{\infty} m_k^+ \mathrm{e}^{-k\Lambda}, \quad M^- = M_0^- - \mathrm{i}m + \sum_{k=1}^{\infty} m_k^- \mathrm{e}^{k\lambda},$$

$$M_0 = t + z\mathrm{e}^{\Lambda} + \bar{z}\mathrm{e}^{-\Lambda} + \dots$$

m, ϕ – real. Reduction: on the circle $\lambda\bar{\lambda} = 1$

$$M^+ = \bar{M}^-,$$
$$\Lambda^+ = -\bar{\Lambda}^-.$$

Lax pair

$$\partial_z \mathbf{\Psi} = L_1 \mathbf{\Psi}, \quad L_1 = (\lambda \mathrm{e}^{\phi} u + v)\partial_t + ((\phi_t v - \phi_z) - \lambda u \mathrm{e}^{\phi}\phi_t)\lambda\partial_\lambda,$$
$$\partial_{\bar z} \mathbf{\Psi} = L_2 \mathbf{\Psi}, \quad L_2 = (\frac{1}{\lambda}\mathrm{e}^{\phi} \bar u + \bar v)\partial_t - ((\phi_t \bar v - \phi_{\bar z}) - \frac{1}{\lambda}\bar u \mathrm{e}^{\phi}\phi_t)\lambda\partial_\lambda.$$

On the unit circle $L_1 = \bar{L}_2$.

$$u = \frac{1}{1 + \mathrm{i} m_t}, \quad v = \frac{\mathrm{i} m_z}{1 - \mathrm{i} m_t}.$$

Equations (for ϕ and m)

$$(v_{\bar z} + \mathrm{e}^{\phi} u \partial_t(\mathrm{e}^{\phi}\bar u) + v\partial_t \bar v) - \text{c.c.} = 0,$$
$$(\partial_{\bar z}(\phi_t v - \phi_z) + \mathrm{e}^{\phi} u \partial_t(\bar u \mathrm{e}^{\phi}\phi_t) - v\partial_t(\phi_t \bar v - \phi_{\bar z}) + u\bar u \mathrm{e}^{2\phi}\phi_t\phi_t)$$
$$+\text{c.c.} = 0$$

If $m = 0$ ($u = 1$, $v = 0$), first equation vanishes, second gives dToda for (-2ϕ).
If $\phi = 0$, second equation vanishes, the first gives

$$(v_{\bar z} + u\partial_t(\bar u) + v\partial_t \bar v) - \text{c.c.} = 0$$

The End