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Two different kinds of rogue waves in weakly crossing sea states

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Formation of giant waves in sea states with two spectral maxima, centered at close wave vectors $\mathbf{k}_0 \pm \Delta\mathbf{k}/2$ in the Fourier plane, is numerically simulated using the fully nonlinear model for long-crested water waves [V. P. Ruban, Phys. Rev. E **71**, 055303(R) (2005)]. Depending on an angle θ between the vectors $\mathbf{k}_0$ and $\Delta\mathbf{k}$, which determines a typical orientation of interference stripes in the physical plane, rogue waves arise having different spatial structure. If $\theta \lesssim \arctan(1/\sqrt{2})$, then typical giant waves are relatively long fragments of essentially two-dimensional (2D) ridges, separated by wide valleys and consisting of alternating oblique crests and troughs. At nearly perpendicular $\mathbf{k}_0$ and $\Delta\mathbf{k}$, the interference minima develop to coherent structures similar to the dark solitons of the nonlinear Schrodinger equation, and a 2D freak wave looks much as a piece of a 1D freak wave, bounded in the transversal direction by two such dark solitons.

V. P. Ruban, Phys. Rev. E. **79**, 065304(R) (2009).

I. Preliminary qualitative remarks

Weakly nonlinear deep-water gravity waves: 2D NLSE for wave envelope

$$Y(x_1, x_2, t) \approx \mathsf{Re}\left[A(x_1, x_2, t)\exp(ik_0x_1 - i\omega_0 t)\right], \tag{1}$$

$$\frac{i}{\omega_0}\frac{\partial A}{\partial t} + \frac{i}{2k_0}\frac{\partial A}{\partial x_1} = \frac{1}{8k_0^2}\left(\frac{\partial^2 A}{\partial x_1^2} - 2\frac{\partial^2 A}{\partial x_2^2}\right) + \frac{k_0^2}{2}|A|^2 A. \tag{2}$$

Simple 1D-reductions

$$A = k_0^{-1}\Psi(\xi, \tau) \tag{3}$$

$$\xi = k_0[(x_1 - V_{\rm gr}t)\cos\theta + x_2\sin\theta], \quad \tau = \omega_0 t, \quad V_{\rm gr} = (\omega_0/2k_0). \tag{4}$$

$$i\Psi_\tau = \frac{1}{4}\left[(1/2)\cos^2\theta - \sin^2\theta\right]\Psi_{\xi\xi} + \frac{1}{2}|\Psi|^2\Psi. \tag{5}$$

Depending on the sign of the dispersion coefficient $D(\theta) = [(1/2)\cos^2\theta - 2\sin^2\theta]$, the dynamics is quite different. For example, in the focusing case (when $D > 0$), the nonlinearity can become saturated with the so-called (bright) solitons,

$$\Psi_{\rm bs} = \frac{s}{\cosh\left[(s/\sqrt{D})(\xi - \xi_0)\right]} \exp(-i\tau s^2/4 + i\phi_0), \tag{6}$$

where s is a wave steepness, and ξ_0, ϕ_0 are arbitrary constants. These weakly nonlinear solutions describe infinitely long wave ridges consisting of alternating oblique crests and troughs. In a more accurate model for fully nonlinear long-crested deep-water waves, as discussed below, these solutions exist for a long time, before qualitative modifications, in a range $0 < s \lesssim 0.24...0.27$ (depending on θ). In particular, if $\theta = 0$, in the highly nonlinear case $s = 0.20...0.27$ we have here the so called 1D GIANT BREATHERS (Dyachenko, Zakharov).

In the defocusing case (when $D < 0$), the so-called dark solitons are possible,

$$\Psi_{\rm ds} = s\tanh\left[(s/\sqrt{-D})(\xi - \xi_0)\right] \exp(-i\tau s^2/2 + i\phi_0), \tag{7}$$

which separate two domains of opposite amplitude.

In view of the above, it is clear that since the effective dispersion coefficient $D(\theta)$ changes the sign at $\theta_* = \arctan(1/\sqrt{2})$, in the full 2D dynamics of random wave fields there should be two substantially different regimes, one regime at $\theta \lesssim \theta_*$ and another at θ close to $\pi/2$. This hypothesis is confirmed in general by numerical experiments reported here.

II. More accurate model

1. Conformal variables in 3D

$$Z = X + iY = z(u,q,t) = u + (i - \hat{H})Y(u,q,t) \tag{8}$$

$$\hat{H}Y(u,q,t) = \int [i\,\mathsf{sign}\,k] Y_{km}(t) e^{iku+imq} dk\,dm/(2\pi)^2 \tag{9}$$

$$Z_t = iZ_u(1 + i\hat{H}) \left[\frac{(\delta\mathcal{K}/\delta\psi)}{|Z_u|^2}\right], \tag{10}$$

$$\begin{aligned} \psi_t &= -g\,\mathsf{Im}\,Z - \psi_u \hat{H}\left[\frac{(\delta\mathcal{K}/\delta\psi)}{|Z_u|^2}\right] \\ &+ \frac{\mathsf{Im}\Big((1 - i\hat{H})\,[2(\delta\mathcal{K}/\delta Z)Z_u + (\delta\mathcal{K}/\delta\psi)\psi_u]\Big)}{|Z_u|^2}. \end{aligned} \tag{11}$$

2. Approximate kinetic energy functional

$$\mathcal{K} \approx \tilde{\mathcal{K}} = -\frac{1}{2}\int \psi \hat{H} \psi_u \, du \, dq + \tilde{\mathcal{F}}, \qquad (12)$$

$$\begin{aligned}\tilde{\mathcal{F}} \; &= \; \frac{i}{8}\int (Z_u\Psi_q - Z_q\Psi_u)\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)}\, du\, dq \\ &+ \; \frac{i}{16}\int \Big\{ \left[(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u\right] \; \hat{E}\overline{(Z-u)} \\ &\qquad -(Z-u)\, \hat{E}\overline{\left[(Z_u\Psi_q - Z_q\Psi_u)^2/Z_u\right]} \Big\}\, du\, dq. \end{aligned} \qquad (13)$$

$$\Psi \equiv (1 + i\hat{H})\psi$$

$$G(k,m) = \frac{-2i}{\sqrt{k^2+m^2}+|k|}, \qquad (14)$$

$$E(k,m) = \frac{2|k|}{\sqrt{k^2+m^2}+|k|}. \qquad (15)$$

3. Variational derivatives

$$\frac{\delta\tilde{\mathcal{K}}}{\delta\psi} = -\hat{H}\psi_u + 2\,\mathsf{Re}\,\left[(1 - i\hat{H})\frac{\delta\tilde{\mathcal{F}}}{\delta\Psi}\right], \tag{16}$$

$$\begin{aligned}\frac{\delta\tilde{\mathcal{F}}}{\delta\Psi} \;=\; & \frac{i}{8}Z_q\hat{\partial}_u[\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)} + (\Psi_q - Z_q\Psi_u/Z_u)\hat{E}\overline{(Z-u)}] \\ & -\frac{i}{8}\,Z_u\,\hat{\partial}_q[\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)} + (\Psi_q - Z_q\Psi_u/Z_u)\hat{E}\overline{(Z-u)}],\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{\delta\tilde{\mathcal{F}}}{\delta Z} \;=\; & -\frac{i}{8}\Psi_q\hat{\partial}_u[\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)} + (\Psi_q - Z_q\Psi_u/Z_u)\hat{E}\overline{(Z-u)}] \\ & +\frac{i}{8}\,\Psi_u\,\hat{\partial}_q[\hat{G}\overline{(Z_u\Psi_q - Z_q\Psi_u)} + (\Psi_q - Z_q\Psi_u/Z_u)\hat{E}\overline{(Z-u)}] \\ & +\frac{i}{16}\,[\hat{\partial}_u[(\Psi_q - Z_q\Psi_u/Z_u)^2\hat{E}\overline{(Z-u)}] - \hat{E}\overline{(\Psi_q - Z_q\Psi_u/Z_u)^2 Z_u}].\end{aligned} \tag{18}$$

III. Numerical experiments

1. Example of evolution of a perturbed giant breather in 2D.

2. Example of evolution of a perturbed high-amplitude oblique soliton into a zigzag structure

3-4. Two small sets of typical numerical experiments designated as A1-A4 and B1-B3. Within each set, at $t = 0$ the normal Fourier modes of the wave field were taken in the form $a_{km}(0) = cF(k,m)\exp(i\gamma_{km})$, with a positive function $F(k,m)$ having two nearly Gaussian maxima at $\mathbf{k}_0 \pm \Delta\mathbf{k}/2$, and with quasi-random initial phases γ_{km}, different for A and for B. In each experiment a choice of the coefficient c gave different values of the total energy $E_{A1}, E_{A2}, E_{A3}, E_{A4}$ and E_{B1}, E_{B2}, E_{B3}. In set A we took $\mathbf{k}_0 = (40.0, -2.5)$ and $\Delta\mathbf{k} = (7.0, 2.0)$, so a case $\theta < \theta_*$ was simulated, while in set B it was a crossing sea state with $\theta = \pi/2$: $\mathbf{k}_0 \pm \Delta\mathbf{k}/2 = (39.5, \pm 3.5)$.

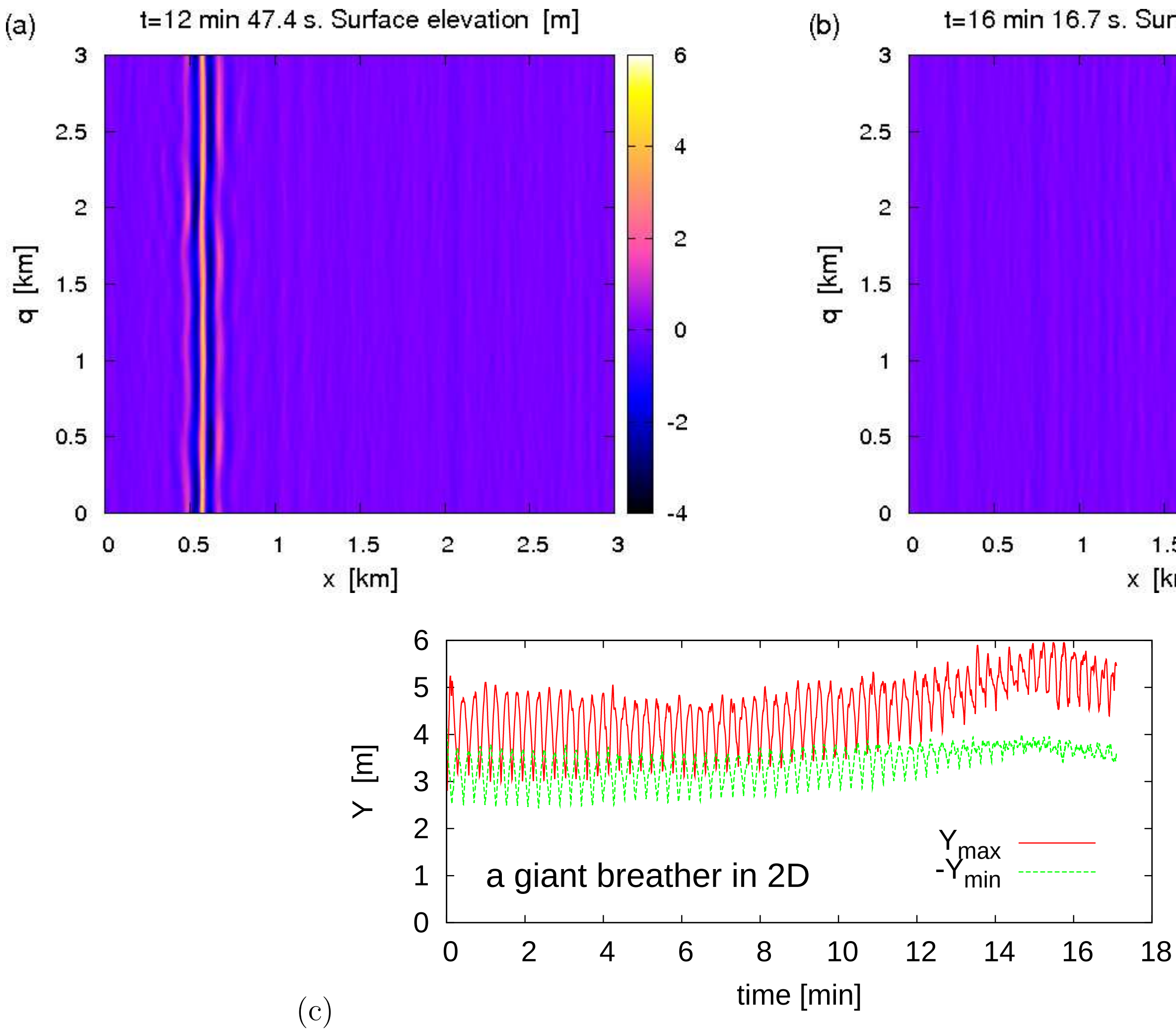

Figure 1: (a)-(b): Evolution of a perturbed giant breather in 2D; (c) Maximum and minimum elevation of the giant breather vs. time.

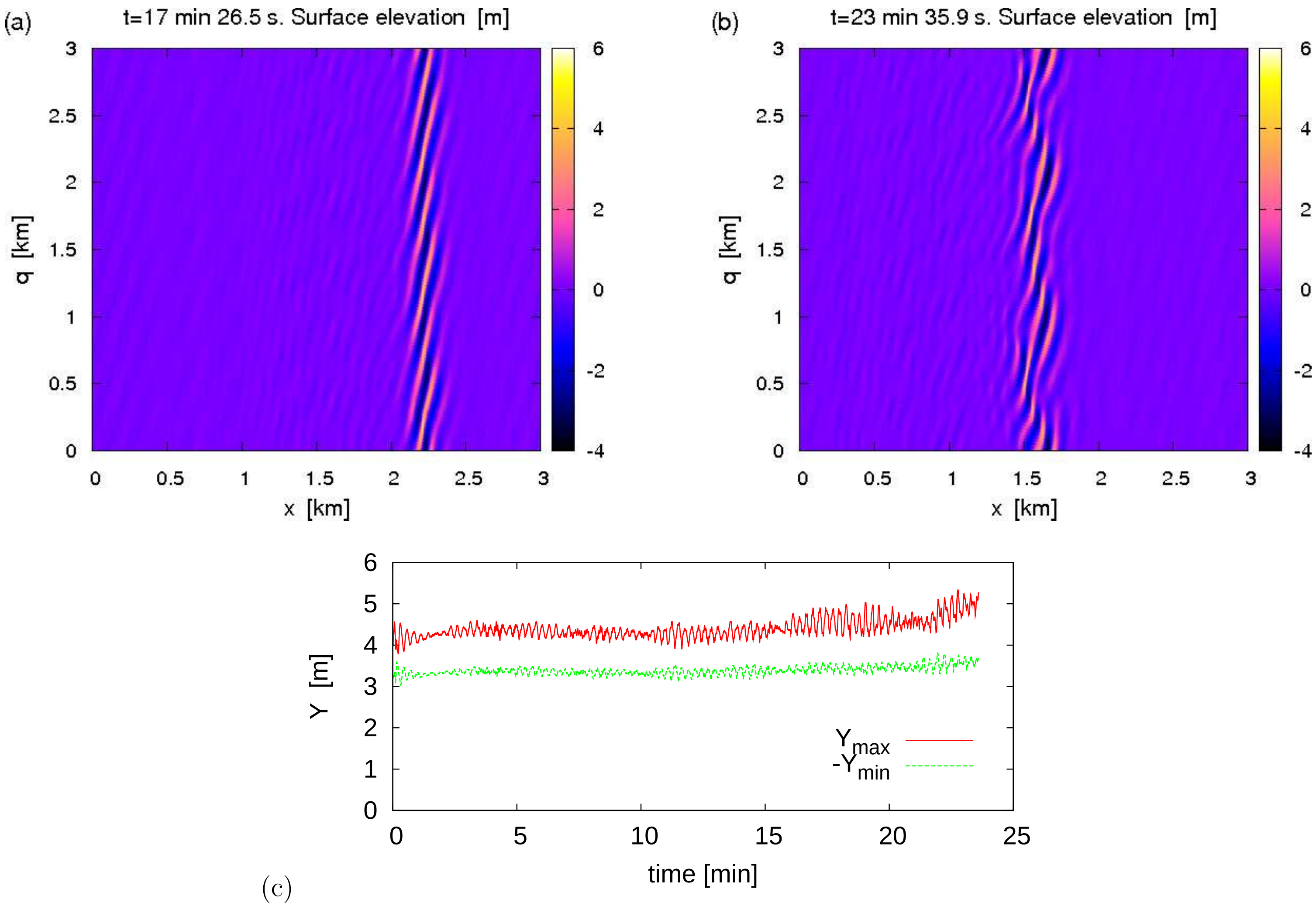

Figure 2: (a)-(b): Evolution of a perturbed high-amplitude oblique soliton into a zigzag structure; (c) Maximum and minimum elevation of the oblique soliton vs. time.

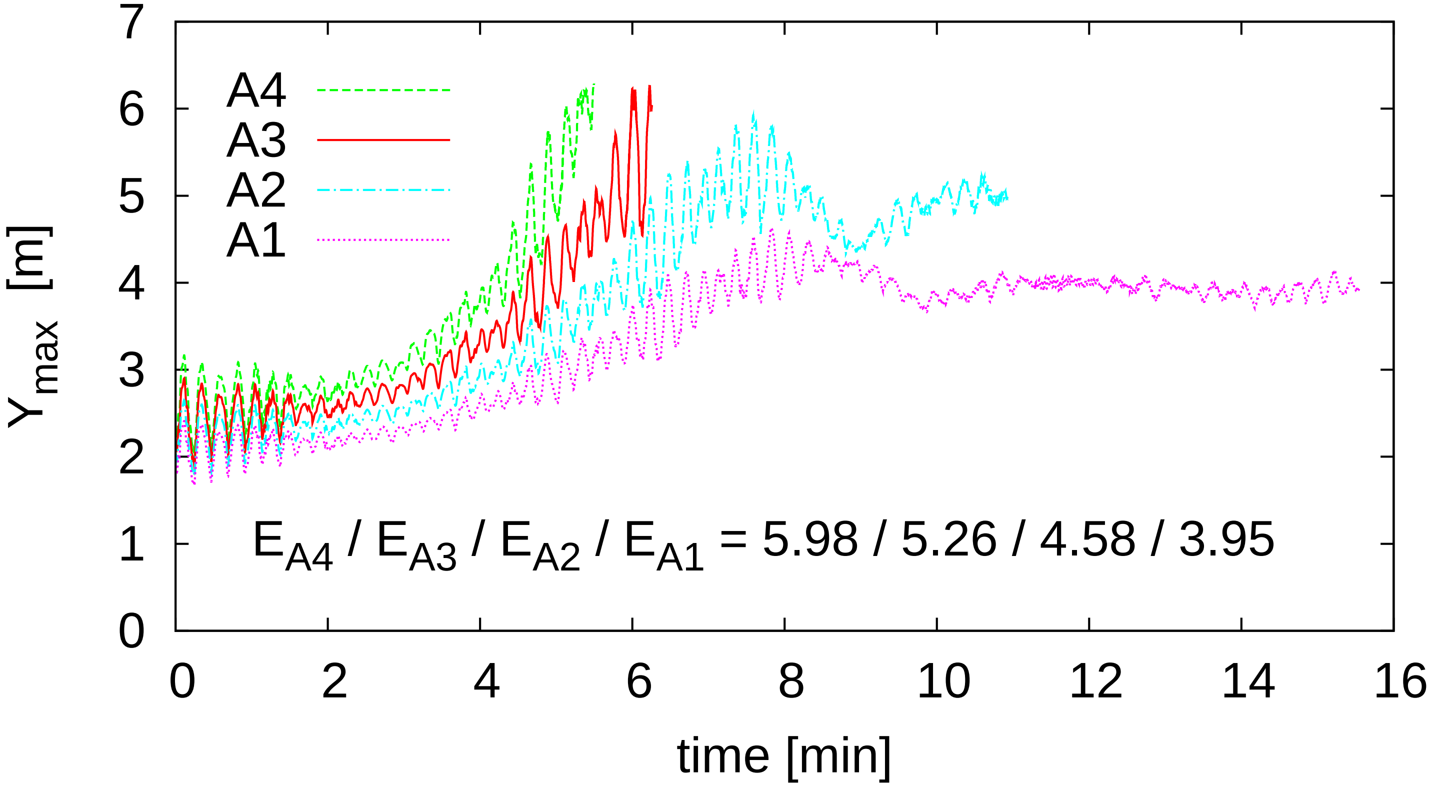

Figure 3: Maximum elevation of the free surface vs. time in the numerical experiments A1-A4.

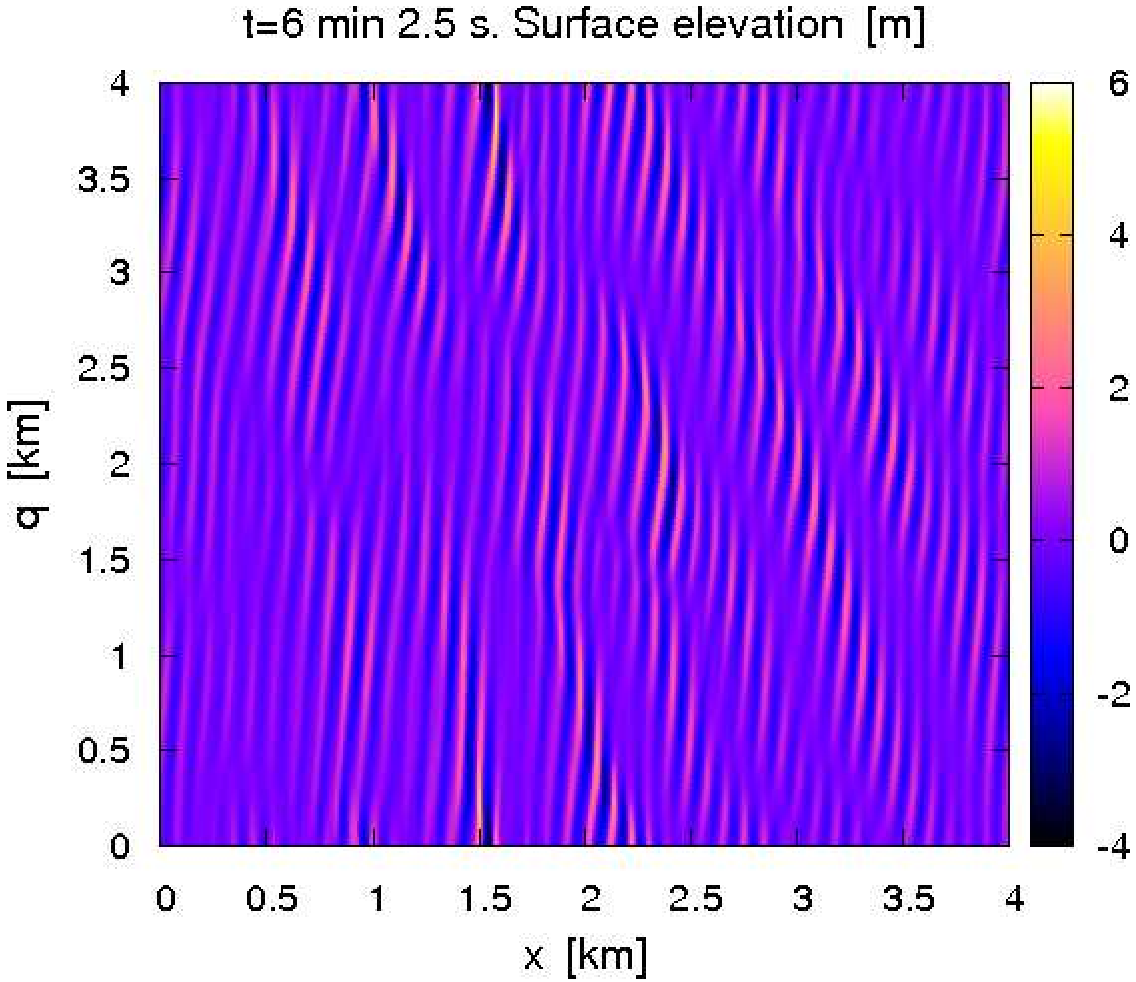

Figure 4: Experiment A3: the two big waves are at $x \approx 1.6$ km, $q \approx [3.7 \cdots 3.9]$ km, and at $x \approx 1.5$ km, $q \approx [0.1 \cdots 0.3]$ km.

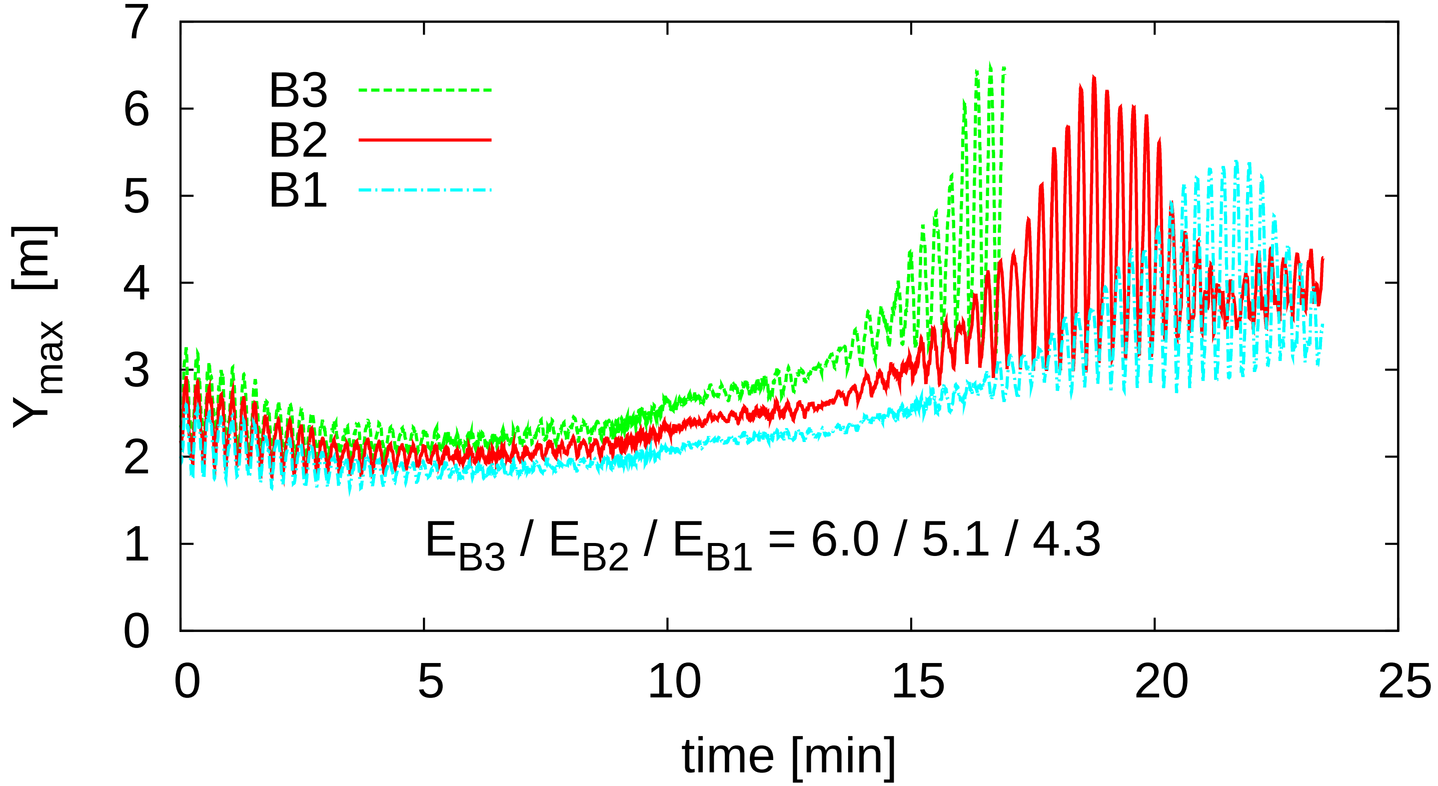

Figure 5: Maximum elevation of the free surface in the numerical experiments B1-B3.

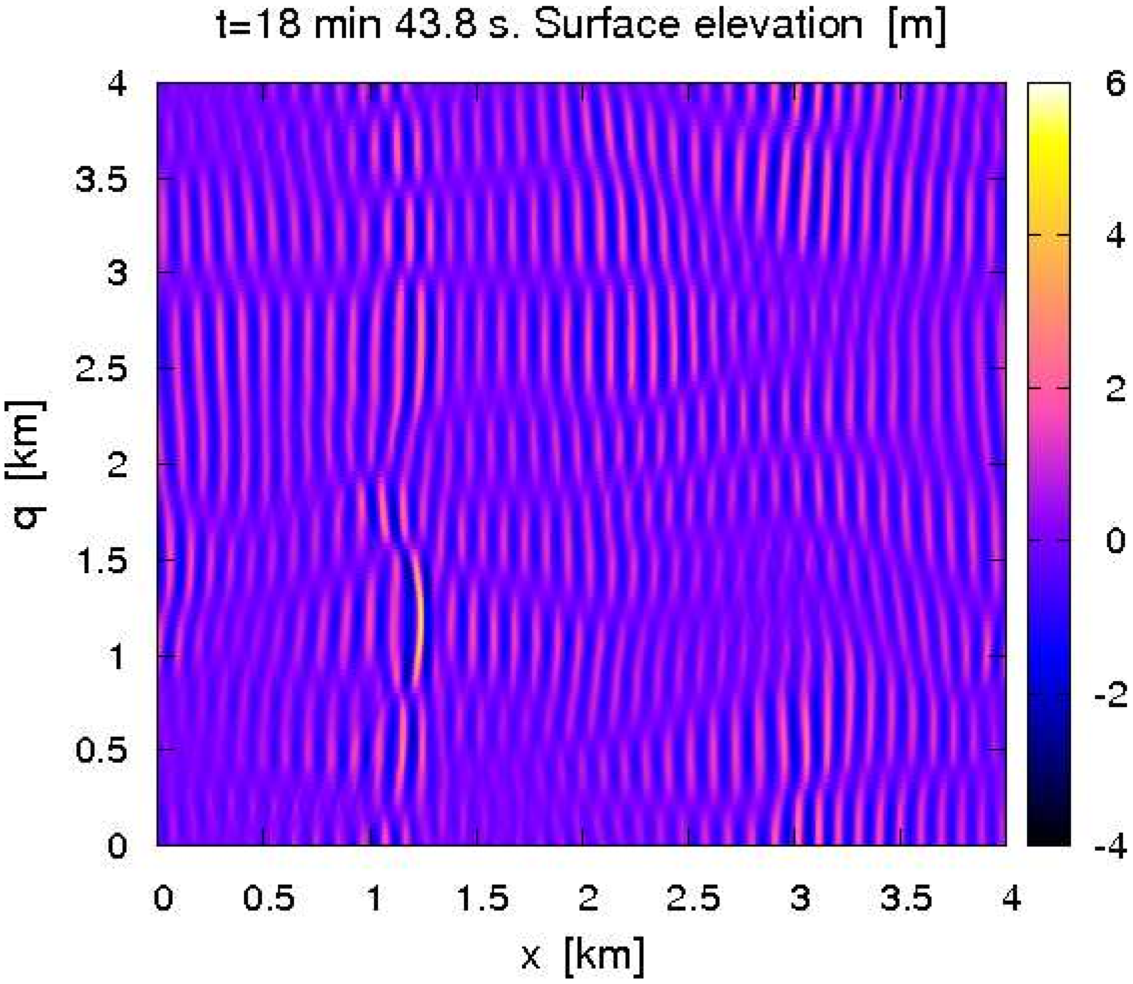

Figure 6: Experiment B2: the rogue wave is at $x \approx 1.2$ km, $q \approx [1.0 \cdots 1.3]$ km.

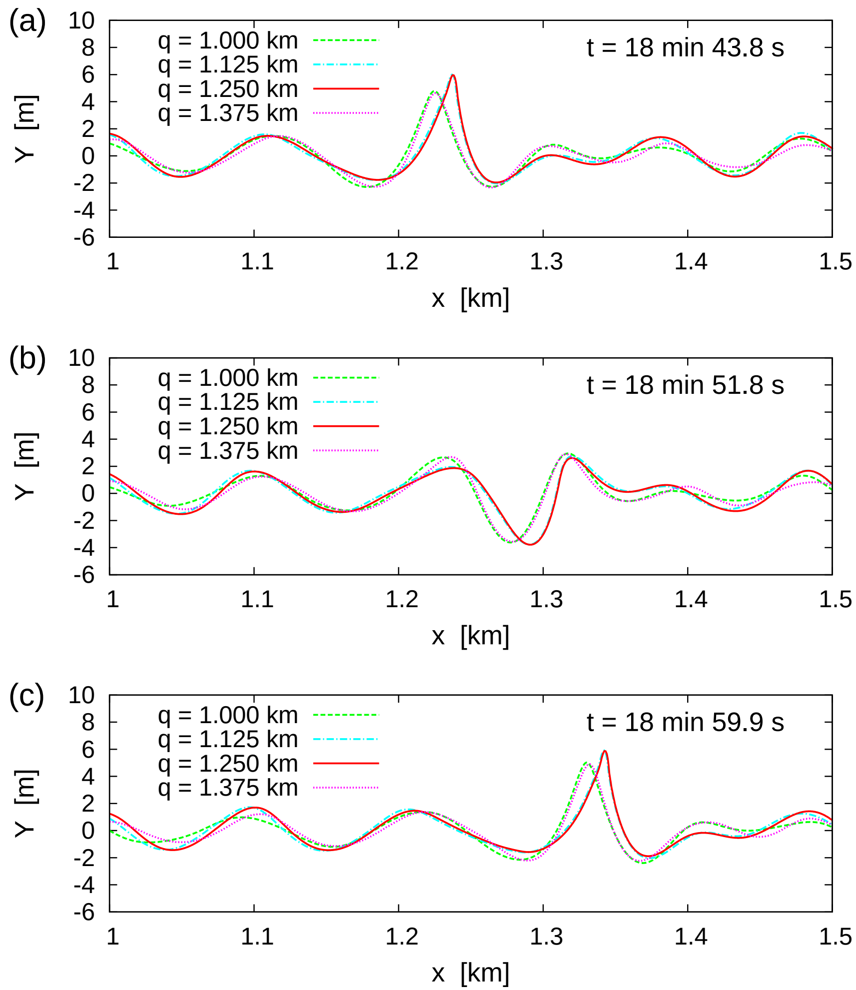

Figure 7: (a) profiles of the freak wave from Fig.6; (b) 8 s later: "a hole in the sea"; (c) 16 s later: the big wave has risen again.