Integrable 3D-systems of hydrodynamic type.

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References

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2. Odesskii A.V. and Sokolov V. V. *Integrable elliptic pseudopotentials*, arXiv:0810.3879, to be published Theor. and Math. Phys

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We consider 3D-systems of the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,y} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,x} = 0,$$

where $i = 1, ..., l$. Here $l \geq n$,

$$\mathbf{u} = (u_1, ..., u_n)^t.$$

Integer $k = l - n$ is called the *defect* of the system.

Equations of the form

$$A_1 Z_{tt} + A_2 Z_{xt} + A_3 Z_{yt} + A_4 Z_{yy} + A_5 Z_{xy} + A_6 Z_{xx} = 0$$

where $A_i = A_i(Z_x, Z_y, Z_t)$, correspond to $n = 3, l = 4$.

Equations

$$F(Z_{tt}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{xy}, Z_{xx}) = 0$$

correspond to $n = 5, l = 8$.

Part 1. Gibbons-Tsarev type systems

The GT-systems play a crucial role in the approach to integrability based on the hydrodynamic reductions.

Definition. A compatible system of PDEs of the form

$$\partial_i p_j = f(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \quad , \; i \neq j, \; i, j = 1, ..., N,$$

$$\partial_i u_k = g_k(p_i, u_1, ..., u_n)\, \partial_i u_1, \qquad k = 2, ..., n, \; i = 1, ..., N,$$

$$\partial_i \partial_j u_1 = h(p_i, p_j, u_1, ..., u_n)\, \partial_i u_1 \partial_j u_1, \qquad i \neq j, \; i, j = 1, ..., N$$

is called *n-fields GT-system*. Here $p_1, ..., p_N$, $u_1, ..., u_n$ are functions of $r^1, ..., r^N$, $N \geq 3$ and $\partial_i = \frac{\partial}{\partial_{r^i}}$.

Definition. Two GT-systems are called *equivalent* if they are related by a transformation of the form

$$p_i \to \lambda(p_i, u_1, ..., u_n), \qquad i = 1, ..., N, \qquad (1)$$

$$u_k \to \mu_k(u_1, ..., u_n), \qquad k = 1, ..., n. \qquad (2)$$

Example 1. The system

$$\partial_i p_j = 0, \qquad \partial_i u_k = g_k(p_i)\partial_i u_1, \qquad \partial_i\partial_j u_1 = 0 \quad (3)$$

is a n-field GT-system for any n, N and any functions $g_k(x)$.

Example 2. Let $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$. Then

$$\partial_{ij}u = \frac{K_2(p_i,p_j)u^2 + K_1(p_i,p_j)u + K_0(p_i,p_j)}{P(u)(p_i-p_j)^2}\,\partial_i u\partial_j u,$$

$$\partial_i p_j = \frac{P(p_j)(u-p_i)}{P(u)(p_i-p_j)}\partial_i u, \quad i,j = 1,...,N, \quad i \neq j,$$

where

$$K_2(p_i,p_j) = 2a_3(p_i-p_j)^2,$$

$$K_1(p_i,p_j) = -a_3(p_i^2p_j+p_ip_j^2)+a_2(p_i^2+p_j^2-4p_ip_j)-a_1(p_i+p_j)-2a_0,$$

$$K_0(p_i,p_j) = 2a_3p_i^2p_j^2+a_2(p_i^2p_j+p_ip_j^2)+a_1(p_i^2+p_j^2)+a_0(p_i+p_j)$$

is an one-field GT-system.

Using transformations of the form

$$u \to \frac{au+b}{cu+d}, \qquad p_i \to \frac{ap_i+b}{cp_i+d},$$

one can put the polynomial P to one of the canonical forms: $P(x) = x(x-1)$, $P(x) = x$, or $P(x) = 1$.

Suppose we have an one-field GT system

$$\partial_i p_j = f(p_i, p_j, u)\, \partial_i u, \qquad \partial_i \partial_j u = h(p_i, p_j, u).$$

We can add one field more to the system as follows:

$$\partial_i v = f(p_i, v, u)\, \partial_i u.$$

We call this procedure *regular field extension*.

Example 3. Let

$$\theta(z,\tau)=\sum_{\alpha\in\mathbb{Z}}(-1)^{\alpha}e^{2\pi i(\alpha z+\frac{\alpha(\alpha-1)}{2}\tau)},\qquad \rho(z,\tau)=\frac{\theta_z}{\theta}.$$

Then

$$\partial_\alpha p_\beta=\frac{1}{2\pi i}(\rho(p_\alpha-p_\beta)-\rho(p_\alpha))\partial_\alpha\tau,$$

$$\partial_\alpha\partial_\beta\tau=-\frac{1}{\pi i}\rho'(p_\alpha-p_\beta)\partial_\alpha\tau\partial_\beta\tau,$$

where $\alpha,\beta=1,...,N,\quad \alpha\neq\beta$, is an one-field GT-system. Regular extensions give rise to

$$\partial_\alpha u_\beta=\frac{1}{2\pi i}(\rho(p_\alpha-u_\beta)-\rho(p_\alpha))\partial_\alpha\tau,\qquad \beta=1,...,n.$$

Another basic notion of the hydrodynamic reduction approach is the *generating relation for reductions*:

$$\frac{\partial_i F(p_j)}{F(p_i) - F(p_j)} = \frac{\partial_i G(p_j)}{G(p_i) - G(p_j)} \qquad (4)$$

Here we omit arguments $u_1, ..., u_n$ in $F,\ G$.

The derivatives in (14) supposed to be calculated in virtue of the GT-system.

For Example 2 with $P(x) = x(x-1)$ there are following n-field solutions (F, G):

Consider the following system of linear PDEs:

$$\frac{\partial^2 h}{\partial u_j \partial u_k} = \frac{s_j}{u_j - u_k} \cdot \frac{\partial h}{\partial u_k} + \frac{s_k}{u_k - u_j} \cdot \frac{\partial h}{\partial u_j}, \quad i, j = 1, ..., n, \quad j \neq k,$$

and

$$\frac{\partial^2 h}{\partial u_j \partial u_j} = -\left(1 + \sum_{k=1}^{n+2} s_k\right) \frac{s_j}{u_j(u_j - 1)} \cdot h +$$

$$\frac{s_j}{u_j(u_j - 1)} \sum_{k \neq j}^{n} \frac{u_k(u_k - 1)}{u_k - u_j} \cdot \frac{\partial h}{\partial u_k} +$$

$$\left(\sum_{k \neq j}^{n} \frac{s_k}{u_j - u_k} + \frac{s_j + s_{n+1}}{u_j} + \frac{s_j + s_{n+2}}{u_j - 1}\right) \cdot \frac{\partial h}{\partial u_j}$$

It is easy to show that the vector space $\mathcal{H}$ of all solutions is $n+1$-dimensional.

For any $h \in \mathcal{H}$ we put

$$S(h,p) = \sum_{1\le i\le n} u_i(u_i-1)(p-u_1)...\widehat{i}...(p-u_n)h_{u_i} +$$

$$(1+\sum_{1\le i\le n+2} s_i)(p-u_1)...(p-u_n)h$$

This is a polynomial of degree n.

Proposition. Let h_1, h_2, h_3 are linearly independent elements of $\mathcal{H}$. Then

$$F = \frac{S(h_1,p)}{S(h_3,p)}, \qquad G = \frac{S(h_2,p)}{S(h_3,p)}$$

satisfy the defining relation for reductions.

In the elliptic case

$$S(h,p) = \sum_{1\le\alpha\le n} \frac{\theta(u_\alpha)\theta(p-u_\alpha-\eta)}{\theta(u_\alpha+\eta)\theta(p-u_\alpha)} h_{u_\alpha} -$$

$$(s_1+...+s_n)\frac{\theta'(0)\theta(p-\eta)}{\theta(\eta)\theta(p)} h.$$

Here $\eta = s_1u_1+...+s_nu_n+r\tau+\eta_0$, where $s_1,...,s_n,r,\eta_0$ are arbitrary constants and $h(u_1,...,u_n,\tau)$ is a solution of the following elliptic hypergeometric system:

$$h_{u_\alpha u_\beta} = s_\beta(\rho(u_\beta - u_\alpha) + \rho(u_\alpha + \eta) - \rho(u_\beta) - \rho(\eta))h_{u_\alpha} +$$

$$s_\alpha(\rho(u_\alpha - u_\beta) + \rho(u_\beta + \eta) - \rho(u_\alpha) - \rho(\eta))h_{u_\beta},$$

$$h_{u_\alpha u_\alpha} = s_\alpha \sum_{\beta \neq \alpha} (\rho(u_\alpha) + \rho(\eta) - \rho(u_\alpha - u_\beta) - \rho(u_\beta + \eta))h_{u_\beta} +$$

$$(\sum_{\beta \neq \alpha} s_\beta \rho(u_\alpha - u_\beta) + (s_\alpha + 1)\rho(u_\alpha + \eta) +$$

$$s_\alpha \rho(-\eta) + (s_0 - s_\alpha - 1)\rho(u_\alpha) + 2\pi i r)h_{u_\alpha} -$$

$$s_0 s_\alpha(\rho'(u_\alpha) - \rho'(\eta))h,$$

$$h_\tau = \frac{1}{2\pi i} \sum_\beta (\rho(u_\beta + \eta) - \rho(\eta))h_{u_\beta} - \frac{s_0}{2\pi i}\rho'(\eta)h.$$

Given GT-system and a solution (F, G) of the defining relation for reduction, one can easily construct an integrable system of the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,y} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,x} = 0,$$

where $i = 1, ..., l$. Here $l \geq n$,

$$\mathbf{u} = (u_1, ..., u_n)^t.$$

Integer $k = l - n$ is called the *defect* of the system.

The coefficients are defined by relations:

$$\sum_{j=1}^{n} (a_{ij}(\mathbf{u}) F(p, u_1, ..., u_n) + b_{ij}(\mathbf{u}) G(p, u_1, ..., u_n) +$$

$$c_{ij}(\mathbf{u})) g_j(p, u_1, ..., u_n) = 0, \qquad i = 1, ..., l,$$

where by definition $g_1 = 1$.

Namely, consider the linear space V of functions in p generated by

$$\{F(p, u_1, ..., u_n) g_j(p, u_1, ..., u_n),$$

$$G(p, u_1, ..., u_n) g_j(p, u_1, ..., u_n),$$

$$g_j(p, u_1, ..., u_n); \quad j = 1, ..., n\}.$$

Then the system contains of l equations iff V is $(3n - l)$-dimensional.

Part 2. Weakly nonlinear systems

For the generic GT-systems the functions f, h have poles at $p_i = p_j$. However, there exist GT-systems holomorphic at $p_i = p_j$.

We call integrable 3D-system related to a GT-system holomorphic at $p_i = p_j$ *weakly nonlinear*. It is possible to check that if $l = n$ then any 2D-system that describe travel wave solutions

$$\mathbf{u} = \mathbf{u}(k_1 x + k_2 y + k_3 t,\ k_4 x + k_5 y + k_6 t)$$

for weakly nonlinear 3D-system is a weakly nonlinear 2D-system.

Example. Consider the following 3D-system (Ferapontov, Khusnutdinova):

$$v_t + av_x + pv_y + qw_y = 0, \quad w_t + bw_x + rv_y + sw_y = 0,$$

where

$$a = w, \quad b = v,$$

$$s = \frac{P(v)}{w - v} + \frac{1}{3}P'(v), \quad p = \frac{P(w)}{v - w} + \frac{1}{3}P'(w),$$

$$r = \frac{P(w)}{w - v}, \quad q = \frac{P(v)}{v - w}.$$

Here P is arbitrary polynomial of third degree.

The corresponding GT-system is given by

$$\partial_1 p_2 = \frac{P(w)}{(w-v)P(v)} p_2^2 p_1 + \left(\frac{1}{w-v} + \frac{P'(v)}{P(v)} \right) p_2 p_1 -$$

$$\left(\frac{1}{v-w} + \frac{P'(w)}{P(w)} \right) p_2 - \frac{P(v)}{(v-w)P(w)},$$

$$\partial_1 v = p_1 \, \partial_1 w,$$

$$\partial_1 \partial_2 w = \left(\frac{P(w)}{(v-w)P(v)} p_1 p_2 + \frac{1}{v-w} + \frac{P'(w)}{P(w)} \right) \partial_1 w \partial_2 w.$$

It is possible to verify that this GT-system is equivalent to

$$\partial_i p_j = 0, \qquad \partial_i u_2 = g_2(p_i)\partial_i u_1, \qquad \partial_i\partial_j u_1 = 0,$$

where

$$g_2(p) = \frac{a_2p^2 + a_1p + a_0}{b_2p^2 + b_1p + b_0}.$$

Example. The dispersionless Hirota equation

$$a_1Z_xZ_{yt} + a_2Z_yZ_{xt} + a_3Z_tZ_{xy} = 0, \qquad a_1 + a_2 + a_3 = 0$$

corresponds to a holomorphic GT-system.

Fix pairwise distinct numbers $\lambda_0, \lambda_1, ..., \lambda_n$. Consider the following n+1-field GT-system with fields $u_1, ..., u_n,\ w$.

$$\partial_i p_j = 0, \quad \partial_i u_j = \frac{\lambda_j - \lambda_0}{p_i - \lambda_j} \partial_i u, \quad \partial_i \partial_j u = 0. \tag{5}$$

For any constant $\mathbf{a} = (a_0, a_1, ..., a_n)$ we put

$$S(\mathbf{a}, p) = \frac{a_0}{p - \lambda_0} + \sum_{i=1}^{n} \frac{a_i e^{u_i}}{p - \lambda_i}.$$

Proposition. The functions

$$F = \frac{S(\mathbf{a}_1, p)}{S(\mathbf{a}_3, p)}, \qquad G = \frac{S(\mathbf{a}_2, p)}{S(\mathbf{a}_3, p)}$$

satisfy the defining relation for reductions.

The corresponding 3D-systems have the form:

$$\sum_{1\le j\le n, j\ne i} (a_{2,i}a_{3,j} - a_{2,j}a_{3,i})e^{u_j}\frac{u_{i,t_1} - u_{j,t_1}}{\lambda_i - \lambda_j} +$$

$$(a_{2,i}a_{3,0} - a_{3,i}a_{2,0})\frac{u_{i,t_1}}{\lambda_i - \lambda_0} +$$

$$\sum_{1\le j\le n, j\ne i} (a_{3,i}a_{1,j} - a_{3,j}a_{1,i})e^{u_j}\frac{u_{i,t_2} - u_{j,t_2}}{\lambda_i - \lambda_j} +$$

$$(a_{3,i}a_{1,0} - a_{1,i}a_{3,0})\frac{u_{i,t_2}}{\lambda_i - \lambda_0} +$$

$$\sum_{1\le j\le n, j\ne i} (a_{1,i}a_{2,j} - a_{1,j}a_{2,i})e^{u_j}\frac{u_{i,x} - u_{j,x}}{\lambda_i - \lambda_j} +$$

$$(a_{1,i}a_{2,0} - a_{2,i}a_{1,0})\frac{u_{i,x}}{\lambda_i - \lambda_0} = 0$$

where $i = 1, ..., n$.

Proposition. This system possesses the following pseudopotential representation

$$\psi_{t_1} = \frac{S(\mathbf{a}_1, \xi)}{S(\mathbf{a}_3, \xi)} \psi_x, \qquad \psi_{t_2} = \frac{S(\mathbf{a}_2, \xi)}{S(\mathbf{a}_3, \xi)} \psi_x,$$

where ξ is a spectral parameter.

Equations of the form

$$A_1 Z_{tt} + A_2 Z_{xt} + A_3 Z_{yt} + A_4 Z_{yy} + A_5 Z_{xy} + A_6 Z_{xx} = 0$$

where $A_i = A_i(Z_x, Z_y, Z_t)$, correspond to $n = 3, l = 4$.

Equations

$$F(Z_{tt}, Z_{xt}, Z_{yt}, Z_{yy}, Z_{xy}, Z_{xx}) = 0$$

correspond to $n = 5, l = 8$.

Definition. An (1+1)-dimensional hydrodynamic type system of the form

$$r_t^i = \lambda^i(r^1, ..., r^N)\, r_x^i, \qquad i = 1, ..., N, \tag{6}$$

is called semi-Hamiltonian if the following relation holds

$$\partial_j \frac{\partial_i \lambda^k}{\lambda^i - \lambda^k} = \partial_i \frac{\partial_j \lambda^k}{\lambda^j - \lambda^k}, \qquad i \neq j \neq k, \tag{7}$$

Recall that semi-Hamiltonian systems have infinitly many symmetries and conservation laws of hydrodynamic type.

Definition. A hydrodynamic reduction of the 3D-system is a pair of compatible semi-Hamiltonian hydrodynamic type systems

$$r^i_t = \lambda^i(r^1, ..., r^N)\, r^i_x, \quad r^i_y = \mu^i(r^1, ..., r^N)\, r^i_x, \qquad i = 1, ..., N, \tag{8}$$

and functions $v_1(r^1, ..., r^N), ..., v_n(r^1, ..., r^N)$ such that for each solution of (8) functions

$$u_1 = v_1(r^1, ..., r^N), ..., u_n = v_n(r^1, ..., r^N) \tag{9}$$

are solutions of the 3D-system.

According to [?] a system (**??**) is called integrable if it possess sufficiently many hydrodynamic reductions. Namely, substitute (9) into (**??**), use (8) and equate coefficients at r^l_x to zero. We obtain

$$\sum_{j=1}^{n} a_{ij}(\mathbf{v})\, \partial_l v_j \lambda^l + \sum_{j=1}^{n} b_{ij}(\mathbf{v})\, \partial_l v_j \mu^l + \sum_{j=1}^{n} c_{ij}(\mathbf{v})\, \partial_l v_j = 0, \qquad i = 1, ..., n+k, \tag{10}$$

For each fixed l this is the same linear overdetermined system for $\partial_l v_1, ..., \partial_l v_n$. This linear system must have non-zero solution so all its $n \times n$ minors must be equal to zero. These minors are polynomials in λ^l, μ^l independent on l. We assume that these system of polynomial equations is equivalent to one equation

$$P(\lambda^l, \mu^l) = 0 \qquad (11)$$

(othewise λ^l, μ^l are fixed and we don't have sufficiently many reductions). Equation (11) defines the so-called disspersion curve. Let p be a coondinate on this curve. Then (11) is equivalent to equations

$$\lambda^l = F(p_l, v_1, ..., v_n), \quad \mu^l = G(p_l, v_1, ..., v_n)$$

for some functions F, G. Assume that for generic p_l the linear system (10) has one solution up to proportionality. Solving this system we obtain

$$\partial_i v_k = g_k(p_i, v_1, ..., v_n)\, \partial_i v_1, \qquad k = 2, ..., n,\ i = 1, ..., N, \qquad (12)$$

for some functions g_k. Rewrite (8) in the form

$$r^i_t = F(p_i, v_1, ..., v_n) r^i_x, \quad r^i_y = G(p_i, v_1, ..., v_n) r^i_x, \qquad i = 1, ..., N, \qquad (13)$$

and note that compatibility condition reads

$$\frac{\partial_i F(p_j)}{F(p_i) - F(p_j)} = \frac{\partial_i G(p_j)}{G(p_i) - G(p_j)} \tag{14}$$

Here we omit arguments $v_1, ..., v_n$ in F, G. From (14) we can find $\partial_i p_j$ in the form

$$\partial_i p_j = f(p_i, p_j, v_1, ..., v_n)\, \partial_i v_1 \qquad , \ i \neq j, \ i, j = 1, ..., N.$$

Finally, compatibility condition $\partial_i \partial_j v_k = \partial_j \partial_i v_k$ for some k gives

$$\partial_i \partial_j v_1 = h(p_i, p_j, v_1, ..., v_n)\, \partial_i v_1 \partial_j v_1, \qquad i \neq j, \ i, j = 1, ..., N.$$

Collecting these equations together we obtain a system of the form (**??**). Hydrodynamic reductions of (**??**) depend on solution of this system (**??**). We want to have as many reductions as possible, therefore we assume that the system (**??**) is compatible. In this case hydrodinamic reduction locally depends on N funktions in one variable.

Integrable 3D-systems related to the generalized hypergeometric functions

We construct new wide classes of pseudopotentials written in the following parametric form:

$$\Phi_y = F_1(p, \mathbf{u}), \qquad \Phi_t = F_2(p, \mathbf{u}), \qquad \Phi_x = F_3(p, \mathbf{u}),$$

where $\mathbf{u} = (u_1, ..., u_n)$ and the p-dependence of functions F_i is defined by the ODE

$$F_{i,p} = \phi_i(p, \mathbf{u}) \cdot^{-s_1} (p-1)^{-s_2} (p-u_1)^{-s_3} ... (p-u_n)^{-s_{n+2}}$$

Here $s_1, ..., s_{n+2}$ are arbitrary constants and ϕ_i are some polynomials in p of degree $n-k$.

We call them *pseudopotentials of defect* k.

for unknown function $h(u_1,\dots,u_n)$. If $n=1$, then this system coincides with the standard hypergeometric equation

$$u(u-1)\,y(u)''+[(\alpha+\beta+1)\,u-\gamma]\,y(u)'+\alpha\beta\,y(u)=0,$$

where $s_1=-\alpha,\ s_2=\alpha-\gamma,\ s_3=\gamma-\beta-1$.

Proposition 1. This system is compatible for any constants $s_1,\dots,s_{n+2}$. The dimension of the linear space $\mathcal{H}$ of solutions of the system equals $n+1$.

Define function $P(g,\zeta)$ by

$$P(g,\zeta) = \int_0^\zeta S(g,p)(p-u_1)^{-s_1-1}...(p-u_n)^{-s_n-1}\times$$

$$p^{-s_{n+1}-1}(p-1)^{-s_{n+2}-1}dp.$$

Let $g_0, g_1, g_2 \in \mathcal{H}$ be linear independent.

Theorem. The compatibility conditions $\Phi_{t_i t_j} = \Phi_{t_j t_i}$ for the system

$$\Phi_{t_\alpha} = P(g_\alpha, p), \qquad \alpha = 0,1,2 \tag{15}$$

are equivalent to a system of PDEs for $u_1, ..., u_n$ of the form:

$$\sum_{j=1}^n a_{ij}(\mathbf{u})\, u_{j,t_1} + \sum_{j=1}^n b_{ij}(\mathbf{u})\, u_{j,t_2} + \sum_{j=1}^n c_{ij}(\mathbf{u})\, u_{j,t_0} = 0,$$

where $i = 1, ..., n$, and $t_0 = x$.

The explicit form of this system is given by

$$\sum_{i\neq j}((g_{1,u_j}g_{2,u_i}-g_{2,u_j}g_{1,u_i})\frac{u_j(u_j-1)u_{i,t_0}-u_i(u_i-1)u_{j,t_0}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_1g_{2,u_j}-g_2g_{1,u_j})u_{j,t_0}+$$

$$\sum_{i\neq j}((g_{2,u_j}g_{0,u_i}-g_{0,u_j}g_{2,u_i})\frac{u_j(u_j-1)u_{i,t_1}-u_i(u_i-1)u_{j,t_1}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_2g_{0,u_j}-g_0g_{2,u_j})u_{j,t_1}+$$

$$\sum_{i\neq j}((g_{0,u_j}g_{1,u_i}-g_{1,u_j}g_{0,u_i})\frac{u_j(u_j-1)u_{i,t_2}-u_i(u_i-1)u_{j,t_2}}{u_j-u_i}+$$

$$(1+s_1+...+s_{n+2})(g_0g_{1,u_j}-g_1g_{0,u_j})u_{j,t_2}=0.$$

Pseudopotentials of defect $k > 0$

To define pseudopotentials of defect k, we fix k linearly independent generalized hypergeometric functions $h_1, ..., h_k \in \mathcal{H}$. For any $g \in \mathcal{H}$ define $S_k(g, p)$ by

$$S_k(g,p) = \frac{1}{\Delta} \sum_{1 \le i \le n-k+1} u_i(u_i - 1)(p - u_1) \times ...\hat{i}... \times (p - u_{n-k+1}) \Delta_i(g).$$

Here

$$\Delta = \det \begin{pmatrix} h_1 & ... & h_k \\ h_{1,u_{n-k+2}} & ... & h_{k,u_{n-k+2}} \\ .... & ... & .... \\ h_{1,u_n} & ... & h_{k,u_n} \end{pmatrix},$$

$$\Delta_i(g) = \det \begin{pmatrix} g & h_1 & \dots & h_k \\ g_{u_i} & h_{1,u_i} & \dots & h_{k,u_i} \\ g_{u_{n-k+2}} & h_{1,u_{n-k+2}} & \dots & h_{k,u_{n-k+2}} \\ \dots\dots\dots & \dots & \dots & \dots\dots\dots \\ g_{u_n} & h_{1,u_n} & \dots & h_{k,u_n} \end{pmatrix}.$$

It is clear that $S_{n,k}(g,p)$ is a polynomial in p of degree $n-k$.

Example 3. In the simplest case $n = 2$, $k = 1$ we have

$$S_1(g,p) = u_1(u_1 - 1)(p - u_2)\frac{gh_{1,u_1} - g_{u_1}h_1}{h_1} +$$

$$u_2(u_2 - 1)(p - u_1)\frac{gh_{1,u_2} - g_{u_2}h_1}{h_1}.$$

Define the function $P_k(g,p)$ by

$$P_k(g,p) = \int_0^p S_k(g,p)(p{-}u_1)^{-s_1-1}...(p{-}u_{n-k+1})^{-s_{n-k+1}-1}$$

$$\times(p{-}u_{n-k+2})^{-s_{n-k+2}}...(p{-}u_n)^{-s_n}p^{-s_{n+1}-1}(p{-}1)^{-s_{n+2}-1}dp.$$

Theorem. The compatibility conditions $\Phi_{t_i t_j} = \Phi_{t_j t_i}$ for the system

$$\Phi_{t_\alpha} = P_k(g_\alpha, p), \qquad \alpha = 0,1,2 \tag{16}$$

are equivalent to the following system of PDEs for $u_1, ..., u_n$ of the defect k:

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_q)\Delta_i(g_r) - \Delta_j(g_r)\Delta_i(g_q))$$

$$\times \frac{u_j(u_j-1)u_{i,t_s} - u_i(u_i-1)u_{j,t_s}}{u_j-u_i} +$$

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_r)\Delta_i(g_s) - \Delta_j(g_s)\Delta_i(g_r))$$

$$\times \frac{u_j(u_j-1)u_{i,t_q} - u_i(u_i-1)u_{j,t_q}}{u_j-u_i} +$$

$$\sum_{1\le i\le n-k, i\neq j} (\Delta_j(g_s)\Delta_i(g_q) - \Delta_j(g_q)\Delta_i(g_s))$$

$$\times \frac{u_j(u_j-1)u_{i,t_r} - u_i(u_i-1)u_{j,t_r}}{u_j-u_i} = 0,$$

where $j = 1, ..., n-k$ and

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r)u_{i,t_s} = \sum_{i=1}^{n-k+1} \Delta_i(g_s)u_{i,t_r},$$

$$\sum_{i=1}^{n-k+1} \Delta_i(g_r)\frac{u_m(u_m-1)u_{i,t_s} - u_i(u_i-1)u_{m,t_s}}{u_m - u_i} =$$

$$\sum_{i=1}^{n-k+1} \Delta_i(g_s)\frac{u_m(u_m-1)u_{i,t_r} - u_i(u_i-1)u_{m,t_r}}{u_m - u_i},$$

where $m = n-k+2, ..., n$. Here q, r, s run from 0 to n and $t_0 = x$.

Example 4. In the case $n = 3$, $k = 1$ the formulas can be rewritten as follows. Let h_1, g_0, g_1, g_2 be linearly independent elements of $\mathcal{H}$. Denote by B_{ij} the cofactors of the matrix

$$\begin{pmatrix} h_1 & g_0 & g_1 & g_2 \\ h_{1,u_1} & g_{0,u_1} & g_{1,u_1} & g_{1,u_1} \\ h_{1,u_2} & g_{0,u_2} & g_{1,u_2} & g_{1,u_1} \\ h_{1,u_3} & g_{0,u_3} & g_{1,u_3} & g_{1,u_3} \end{pmatrix}.$$

Define vector fields V_i by

$$V_1 = B_{22}\,\frac{\partial}{\partial t_0} + B_{23}\,\frac{\partial}{\partial t_1} + B_{24}\,\frac{\partial}{\partial t_2},$$

$$V_2 = B_{32}\,\frac{\partial}{\partial t_0} + B_{33}\,\frac{\partial}{\partial t_1} + B_{34}\,\frac{\partial}{\partial t_2},$$

$$V_3 = B_{42}\,\frac{\partial}{\partial t_0} + B_{43}\,\frac{\partial}{\partial t_1} + B_{44}\,\frac{\partial}{\partial t_2}.$$

Then the set of equations is equivalent to

$$V_1(u_2) = V_2(u_1), \quad V_2(u_3) = V_3(u_2), \quad V_3(u_1) = V_1(u_3).$$

and

$$u_3(u_3-1)(u_1-u_2)V_1(u_2) + u_1(u_1-1)(u_2-u_3)V_2(u_3)$$
$$+u_2(u_2-1)(u_3-u_1)V_3(u_1) = 0.$$

There exist conservation laws of the form

$$\left(\frac{g_r}{h_1}\right)_{t_s} = \left(\frac{g_s}{h_1}\right)_{t_r}.$$

Introducing z such that $z_{t_r} = \frac{g_r}{h_1}$, we reduce the system to a quasi-linear equation of the form

$$\sum_{i,j} P_{i,j}(z_{t_0}, z_{t_1}, z_{t_2})\, z_{t_i,t_j} = 0, \qquad i,j = 0,1,2. \quad (17)$$

In the paper by E. Feropontov an inexplicit description of all integrable equations (17) was proposed. The equation constructed above corresponds to the generic case in this classification. Indeed, it depends on five essential parameters $s_1, ..., s_5$ which agrees with the results of this paper.

Integrable elliptic pseudopotentials

If

$$\Phi_t = A(p, \mathbf{u}), \qquad \Phi_y = B(p, \mathbf{u}), \qquad \text{where} \quad p = \Phi_x$$

is a pseudopotential representation for some integrable 3D-system, then for any $p \in \mathbb{C}$ the point $\left(\frac{A_{ppp}}{A_{pp}^2}, A_p\right)$ belongs to an algebraic curve of genus g, whose coefficients depend on $\mathbf{u}$.

Now we construct pseudopotentials and integrable systems related to the elliptic curve. For these systems $\mathbf{u} = (u_1, \dots, u_n, \tau)$, where τ is the parameter of the elliptic curve also being an unknown function in the system.

The coefficients of the systems are expressed in terms of the following elliptic generalization of hypergeometric functions in several variables:

$$g_{u_\alpha u_\beta} = s_\beta(\rho(u_\beta - u_\alpha) + \rho(u_\alpha + \eta) - \rho(u_\beta) - \rho(\eta))g_{u_\alpha} +$$

$$s_\alpha(\rho(u_\alpha - u_\beta) + \rho(u_\beta + \eta) - \rho(u_\alpha) - \rho(\eta))g_{u_\beta},$$

$$g_{u_\alpha u_\alpha} = s_\alpha \sum_{\beta\neq\alpha} (\rho(u_\alpha) + \rho(\eta) - \rho(u_\alpha - u_\beta) - \rho(u_\beta + \eta))g_{u_\beta} +$$

$$(\sum_{\beta\neq\alpha} s_\beta \rho(u_\alpha - u_\beta) + (s_\alpha + 1)\rho(u_\alpha + \eta) +$$

$$s_\alpha \rho(-\eta) + (s_0 - s_\alpha - 1)\rho(u_\alpha) + 2\pi i r)g_{u_\alpha} -$$

$$s_0 s_\alpha(\rho'(u_\alpha) - \rho'(\eta))g,$$

$$g_\tau = \frac{1}{2\pi i}\sum_\beta (\rho(u_\beta + \eta) - \rho(\eta))g_{u_\beta} - \frac{s_0}{2\pi i}\rho'(\eta)g$$

for a single function $g(u_1, \ldots, u_n, \tau)$.

Here $\eta = s_1 u_1 + ... + s_n u_n + r\tau + \eta_0, \quad s_0 = -s_1 - ... - s_n,$
where $s_1, ..., s_n, r, \eta_0$ are arbitrary constants, and

$$\theta(z) = \sum_{\alpha \in \mathbb{Z}} (-1)^\alpha e^{2\pi i (\alpha z + \frac{\alpha(\alpha-1)}{2}\tau)}, \qquad \rho(z) = \frac{\theta'(z)}{\theta(z)}.$$

We omit the second argument τ of the functions θ, ρ and use the notation

$$\rho'(z) = \frac{\partial \rho(z)}{\partial z}, \quad \rho_\tau(z) = \frac{\partial \rho(z)}{\partial \tau}, \quad \theta'(z) = \frac{\partial \theta(z)}{\partial z}, \quad \theta_\tau(z) = \frac{\partial \theta(z)}{\partial \tau}.$$

It turns out that the dimension of the space of solutions for the system equals $n + 1$.

Describe pseudopotentials of defect $k = 0$ related to the elliptic hypergeometric functions. The pseudopotential $A_n(p, u_1, ..., u_n, \tau)$ is defined in a parametric form by

$$A_n = P_n(g_1, p), \qquad p = P_n(g_0, p),$$

where $g_1,\ g_0$ be linearly independent elliptic hypergeometric functions

$$P_n(g, p) = \int_0^p S_n(g, p) e^{2\pi i r(\tau - p)} \times$$

$$\frac{\theta'(0)^{-s_1 - \dots - s_n} \theta(u_1)^{s_1} \dots \theta(u_n)^{s_n}}{\theta(p)^{-s_1 - \dots - s_n} \theta(p - u_1)^{s_1} \dots \theta(p - u_n)^{s_n}} dp,$$

and

$$S_n(g, p) = \sum_{1 \le \alpha \le n} \frac{\theta(u_\alpha)\theta(p - u_\alpha - \eta)}{\theta(u_\alpha + \eta)\theta(p - u_\alpha)} g_{u_\alpha} -$$

$$(s_1 + \dots + s_n) \frac{\theta'(0)\theta(p - \eta)}{\theta(\eta)\theta(p)} g.$$

We call them *elliptic pseudopotential of defect 0*.

Some important examples of pseudopotentials A, B related to the Whitham averaging procedure for integrable dispersion PDEs, to the Frobenious manifolds, and to the WDVV-associativity equation were found by B. Dubrovin and I. Krichever.

In the case $s_1 = ... = s_n = r = 0$, $\eta_0 \to 0$ our pseudopotentials coincide with elliptic pseudopotentials constructed by Dubrovin and Krichever.

Our goal now is to describe "integrable" *pseudopotentials* $A = \psi(p, \mathbf{u})$.

Consider the simplest one-field case: $A = \psi(p, u)$. The Benney hierarchy provides the following two examples

$$\psi = \frac{p^2}{2} + u, \quad \text{and} \quad \psi = \log(p - u).$$

One explicit example more:

$$\psi = \sqrt{u\,(p^2 + c_1) + c_2}.$$

Integrable pseudopotentials in the one-field case

"Integrable" pseudopotentials $\psi(u,p)$ are given by

$$\psi_u = \frac{Q(\psi_p)}{\psi_{pp}}, \qquad \frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{R(\psi_p)}{Q(\psi_p)}, \tag{18}$$

where R and Q are polynomials in ψ_p such that $\deg R \leq 3$, $\deg Q \leq 4$. In the generic case (18) implies

$$\frac{\psi_{ppp}}{\psi_{pp}^2} = \frac{k_1}{\psi_p - b_1} + ... + \frac{k_4}{\psi_p - b_4}, \tag{19}$$

$$b_i' = (1 - k_i)\, a \prod_{j \neq i} (b_i - b_j), \qquad i = 1, ..., 4, \tag{20}$$

where k_i are any constants such that $k_1 + ... + k_4 = 3$, and $b_i = b_i(u)$. The function $a(u)$ can be chosen arbitrarily due to the admissible transformations $u \to s(u)$.

Let us choose

$$a = \frac{1}{(b_2 - b_3)(b_1 - b_4)} + \frac{1}{(b_1 - b_2)(b_3 - b_4)}.$$

Then the general solution of (20) is given by

$$b_1 = \frac{z_2 + u y_2}{z_1 + u y_1}, \quad b_2 = \frac{y_2}{y_1}, \quad b_3 = \frac{z_2 + y_2}{z_1 + y_1}, \quad b_4 = \frac{z_2}{z_1},$$

where $y_i(u)$ are two arbitrary solutions of the gypergeometric equation

$$u(u-1)\, y(u)'' + [(\alpha + \beta + 1)\, u - \gamma]\, y(u)' + \alpha\beta\, y(u) = 0,$$

where $k_1 = 1 + \alpha - \gamma$, $k_2 = 1 - \alpha$, $k_3 = \gamma - \beta$, and

$$z_i = -u y_i + \frac{u(u-1)}{k_1 + k_2 + k_3 - 2}\, y_i'.$$

System (18) can be reduced to quadratures as follows.
Determine $\phi(u,p)$ as the solution of the system:

$$\phi_u = -\frac{\phi(\phi-1)\, y_1'}{\beta(y_1\phi + z_1)}, \qquad \phi_p = \frac{\phi^{k_1}(\phi-u)^{k_2}(\phi-1)^{k_3}}{y_1\phi + z_1}.$$

Then the solution of the following system in involution

$$\psi_u = \frac{y_2 y_1' - y_1 y_2'}{\beta(y_1\phi + z_1)}\, \phi^{1-k_1}(\phi-u)^{1-k_2}(\phi-1)^{1-k_3},$$

$$\psi_p = \frac{y_2\phi + z_2}{y_1\phi + z_1},$$

is a general solution of (18).

Definition of integrability

Consider the dispersionless Lax equation

$$L_t = \{\psi, L\} \tag{21}$$

Suppose there exists a hydrodynamic-type system

$$r_t^i = v^i(\mathbf{r}) r_x^i \qquad i = 1, 2, ..., N, \tag{22}$$

and functions $u = u(\mathbf{r})$ and $L = L(\mathbf{r}, p)$ such that these functions satisfy (21) for any solution $\mathbf{r}(x, t)$ of (26). Then (26) is called *a hydrodynamic reduction* for (21).

The pseudopotential $\psi(u, p)$ is called *integrable* if (21) has "many" hydrodynamic reductions with arbitrary N.

Example. Let us show that $\psi = \ln(p-u)$ is integrable. Let $w(r^1, ..., r^N)$, $p_i(r^1, ..., r^N)$, $i = 1, ..., N$ be an arbitrary solution of the following system

$$\partial_j p_i = \frac{\partial_j w}{p_j - p_i}, \quad \partial_{ij} w = \frac{2\partial_i w \partial_j w}{(p_i - p_j)^2}, \qquad j = 1, ..., N, \quad i \neq j.$$

Here $\partial_i \equiv \frac{\partial}{\partial r^i}$. This system is in involution and therefore its solution depends on $2N$ functions of one variable.

Define a function $L(p, r^1, ..., r^N)$ by

$$\partial_i L = \frac{\partial_i w L_p}{p - p_i}, \quad i = 1, ..., N. \tag{23}$$

The system (23) defines the function L uniquely up to unessential transformations $L \to \lambda(L)$.

Let $u(r^1, ..., r^N)$ be a solution of the system

$$\partial_i u = \frac{\partial_i w}{p_i - u}, \quad i = 1, ..., N. \tag{24}$$

Proposition. The system

$$r^i_t = \frac{1}{p_i - u} r^i_x \tag{25}$$

is a hydrodynamic reduction of (21). ■

Let us introduce the following notation:

$$f_i = \frac{\psi_u}{\psi_p|_{p=p_i} - \psi_p}, \qquad f_{ij} = \frac{\psi_u|_{p=p_j}}{\psi_p|_{p=p_i} - \psi_p|_{p=p_j}}, \quad i \neq j.$$

Theorem. For any integrable pseudopotential $\psi(u,p)$ the following functional equation

$$\partial_p \left(\frac{f_{12}\,\partial_{p_2} f_2 - f_{21}\,\partial_{p_1} f_1 + \partial_u (f_2 - f_1) + f_1\,\partial_p f_2 - f_2\,\partial_p f_1}{f_1 - f_2} \right) = 0$$

holds.

The pseudopotentials described above correspond to the generic solution of this functional equation.

Integrable 2D-systems

The hydrodynamic reductions of our pseudopotentials of defect 0 are integrable systems of the form

$$r^i_t = v^i(r^1, ..., r^N) r^i_x, \qquad i = 1, 2, ..., N. \tag{26}$$

The velocities v^i are defined by an universal overdetermined compatible system of PDEs of the form

$$\partial_i p_j = \frac{p_j(p_j - 1)}{p_i - p_j} \partial_i w, \quad \partial_{ij} w = \frac{2p_i p_j - p_i - p_j}{(p_i - p_j)^2} \partial_i w \, \partial_j w$$

for some functions $w(r^1, ..., r^N)$, $p_i(r^1, ..., r^N)$. Here $i, j = 1, ..., N, i \neq j$.

Define functions u_i by the following system of PDEs

$$\partial_i u_j = \frac{u_j(u_j - 1)\partial_i w}{p_i - u_j}, \quad i = 1, ..., N, \quad j = 1, ..., n.$$

Then our integrable 2D-systems are given by

$$r_t^i = \frac{S(g_1, p_i)}{S(g_2, p_i)} r_x^i,$$

where $g_1, g_2 \in \mathcal{H}$.

For some very special values of parameters s_i these systems are related to the Whitham hierarchies, to the Frobenious manifolds, and to the associativity equation.

Canonical series of conservation laws

The transformation $L(x, t, p) \to p(x, t, L)$ reduces

$$L_t = \{\psi, L\}$$

to the following conservative form

$$p_t = \psi(U, p)_x. \tag{27}$$

Here L plays a role of parameter. If we substitute any expansion of p w.r.t. L into (27), we get an infinite sequence of conservation laws.

For the pseudopotentials above constructed we get

$$P_n(h_2, \zeta)_t = P_n(h_1, \zeta)_x,$$

where

$$\zeta = a_0 + a_1 L + a_2 L^2 + ...$$

Definition. Two integrable pseudopotentials $\psi_1, \ \psi_2$ are called compatible if the system

$$L_{t_1} = \{L, \psi_1\}, \quad L_{t_2} = \{L, \psi_2\}$$

possesses sufficiently many compatible pairs of hydrodynamic reductions

$$r^i_{t_1} = v^i_1(r^1, ..., r^N) r^i_x, \quad r^i_{t_2} = v^i_2(r^1, ..., r^N) r^i_x$$

for each $N \in \mathbb{N}$.

If $\psi_1, \ \psi_2$ are compatible, then $\psi = c_1\psi_1 + c_2\psi_2$ is integrable for all constants c_1, c_2.

Example. The pseudopotentials $\psi_1 = \ln(p - u_1)$ and $\psi_2 = \ln(p - u_2)$ are compatible. Moreover,

$$\psi = c_1 \ln(p - u_1) + ... + c_n \ln(p - u_n)$$

is integrable for each constants $c_1, ..., c_n$.

Proposition. Let $h_1, h_2, h_3 \in \mathcal{H}$ are linear independent. Then pseudopotentials

$$\psi_1 = P_n(h_1, \zeta), \quad \psi_2 = P_n(h_2, \zeta), \quad p = P_n(h_3, \zeta)$$

are compatible.

Proposition. The compatibility conditions for the system

$$L_{t_1} = \{L, \psi_1\}, \quad L_{t_2} = \{L, \psi_2\}$$

are equivalent to a quasilinear system of PDEs of the form

$$\sum_{j=1}^{n} a_{ij}(\mathbf{u})\, u_{j,t_\alpha} + \sum_{j=1}^{n} b_{ij}(\mathbf{u})\, u_{j,t_\beta} + \sum_{j=1}^{n} c_{ij}(\mathbf{u})\, u_{j,t_\gamma} = 0,$$

where $i = 1, ..., n$.