Turbulence in Integrable Systems

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1 Introduction

In this talk we will discuss statistical properties of integrable wave systems. To make the formulation of the problem clear, we start with the focusing Nonlinear Schrodinger equation:

$$i\,\Psi_t + \Psi_{xx} + |\Psi|^2\,\Psi = 0, \quad \infty < x < \infty \tag{1.1}$$

Equation (1.1) is well studied in two cases:

1. $|\Psi| \to 0, \qquad x \to \infty$
In this case the classical Inverse Scattering Method is applicable.

2. Ψ is a quasiperiodic function and the corresponding Lax operator L has only finite number of lacunaes. In this case the solution is formulated in terms of Riemann functions on a certain hyperelliptic algebraic curve.

The connection between these two approaches is not properly traced so far. Let us go outside these two frameworks and assume that in the initial moment of time $t = 0$, function $\Psi = \Psi_0(x)$ is a representative of a certain spatially homogeneous random field such that the correlation function

$$< \Psi_0(x)\,\Psi^*(\lambda + \xi) >= F(\xi) \tag{1.2}$$

do exist. It means that we define a certain probabilistic measure on the class of bounded smooth complex functions $\Psi(x)$. If such measure is fixed, it does not depend on time. For a generic choice of measure, function $F(\xi)$ will change in time, "adjusting" itself to a given measure. But we can try to choose the measure by such a special way, that $F(\xi)$ is invariant in time, and for any value of t get

$$< \Psi(x,\, t)\, \Psi^*(x+\xi, t) >= F(\xi), \qquad \frac{dF}{dt} = 0 \tag{1.3}$$

Such measure is called invariant. Can we do this and how?

Let us reformulate the question in terms of Fourier transforms. Let

$$\Psi(x,\, t) = \int_{-\infty}^{\infty} \Psi(k,t)\, e^{i\,k\,x}\, dk \tag{1.4}$$

For any homogeneous random field

$$< \Psi(k,t)\, \Psi^*(k',t) >= N(k,t)\, \delta(k-k') \tag{1.5}$$

In the initial moment of time

$$< \Psi_0(k)\, \Psi_0^*(k') >= N_0(k)\, \delta(k-k') \tag{1.6}$$

Let us note that

$$F(\xi) = \int_{-\infty}^{\infty} N(k)\, e^{i\,k\,\xi}\, dk$$

Brackets in (1.3)-(1.5) mean averaging over the measure. Can we choose it such that $N(k,t) = N_0(k)$? To approach to the solution of this problem, first we consider the linearized Schrodinger equation

$$i\, \Psi_t + \Psi_{xx} = 0 \tag{1.7}$$

In this case, existence of invariant measure for any $F(\xi)$ is an obvious fact. This measure is Gaussian. It means that all higher correlation functions can be expressed through a special density $N(k)$. For any homogeneous random field

$$
\begin{aligned}
&< \Psi^*(k)\,\Psi^*(k_1)\,\Psi(k_2)\,\Psi(k_3) > = \\
&= N_k\,N_{k_1}\,(\delta_{k-k_2}\,\delta_{k_1-k_3} + \delta_{k-k_3}\,\delta_{k_1-k_2}) + I_{kk_1k_2k_3}\,\delta_{k+k_1-k_2-k_3}
\end{aligned}
\tag{1.8}
$$

Here $I_{kk_1k_2k_3}$ is a cumulant. For a Gaussian field the cumulant is zero. It is clear that for Nonlinear Schrodinger equation the invariant measure must be non-Gaussian. Can we construct the cumulant in the forth-order correlation function (1.7) and all higher order cumulants as series in power of N_k? The answer is positive. In the first order of nonlinearity

$$
\begin{aligned}
I_{kk_1k_2k_3} &= 2\,\frac{R(kk_1k_2k_3)}{\Delta(kk_1k_2k_3)} \\
R_{kk_1k_2k_3} &= N_{k_1}\,N_{k_2}\,N_{k_3} + N_k\,N_{k_2}\,N_{k_3} - N_k\,N_{k_1}\,N_{k_2} - N_k\,N_{k_1}\,N_{k_3} \\
\Delta_{kk_1k_2k_3} &= k^2 + k_1^2 - k_2^2 - k_3^2
\end{aligned}
\tag{1.9}
$$

The denominator in (1.9) is zero if

$$
k + k_2, \quad k_1 = k_3 \quad or \quad k = k_3, \quad k_1 = k_2 \tag{1.10}
$$

However, the nominator on the manifold is zero also. It is announced that this process can be confirmed to infinity. All cumulants could be found: all of them are finite and real as (1.9).

Certainly, this is a consequence of integrability of the NSLE. The same statement is correct for all equations of focusing and defocusing NSLE hierarchy, as well as for equations that belong to the KdV hierarchy. However, for three-wave resonant system this nice and elegant statement fails! In a sense it behaves like a non-integrable system.

In non-integrable weakly nonlinear systems, the spectral function $N(k,t)$ depends on time obeying the kinetic equation

$$\frac{dN}{dt} = Snl \tag{1.11}$$

and all invariant measures are generated by stationary spectra, which are solutions of equation

$$Snl = 0 \tag{1.12}$$

The same might happen with an integrable system. As a result, the integrable systems are separated in two essentially different classes: strongly and weakly integrable.

The strongly integrable systems are similar to NLSE. They have infinite amount of invariant measures preserving all arbitrary spectral functions. All collision terms in the wave kinetic equations are cancelled in any order. Moreover, they have one more fundamental property.

Let us study equation (1.1) in the class of fast decaying functions and tend time to $\pm\infty$. The Fourier transform will tend to some limiting values

$$\Psi(k) \to \Psi^{\pm}(k)$$

It is easy to prove that

$$\left|\Psi^{+}(k)\right|^{2}=\left|\Psi^{-}(k)\right|^{2} \tag{1.13}$$

A similar statement is correct for all strongly integrable systems.

All other systems are weakly integrable. The simplest example is a three-wave resonant system. In this case scattering is nontrivial and asymptotic squared amplitudes of the fields do not coincide. The three-wave kinetic equation is nontrivial. The system still has infinite amount of invariant measures, but they are parameterized by functions of only one variable.

The difference between strongly and weakly integrable systems is pretty delicate. For instance, KP-2 equation is a strongly integrable system, while KP-1 equation is only weakly integrable. Thereafter we demonstrate difference between weakly and strongly integrable systems on some basic examples.

2 Statistical description of weakly nonlinear systems

We will discuss the weakly nonlinear wave systems homogenous in space. There is a standard way to develop statistical description of such systems that leads to kinetic equation for waves. First, we start from the following question: what happens with kinetic equation, if the primitive dynamic equations are in some sense "integrable"? Let us study the following dynamic equation:

$$\frac{\partial \Psi_i(k)}{\partial t} = i \frac{\delta H}{\delta \Psi_i^*(k)} \quad i = 1, \ldots N \tag{2.1}$$

Here H is a Hamiltonian and k belongs to K-space, which is different for different systems. The dimension of this space $d = 1, 2$. We can consider several examples.

1.

$$\begin{aligned} H &= H_2 + H_4 \quad N = 1 \\ H_2 &= \int \omega(k) |\Psi_k|^2 dk \\ H_4 &= \frac{1}{2} \int T_{kk_1k_2k_3} \Psi_k^* \Psi_{k_1}^* \Psi_{k_2} \Psi_{k_3} \, \delta_{k+k_1+k_2+k_3} dk \, dk_1 \, dk_2 \, dk_3 \end{aligned} \tag{2.2}$$

In this case equation (2.1) reads:

$$\frac{\partial \Psi}{\partial t} = i \, \omega(k) \, \Psi_k + i \int T_{k\,k_1\,k_2\,k_3} \, \Psi_{k_1}^* \Psi_{k_2} \Psi_{k_3} \, \delta_{k+k_1-k_2-k_3} \, dk \, dk_1 \, dk_2 \, dk_3 \tag{2.3}$$

Term $T_{k\,k_1\,k_2\,k_3}$ satisfies symmetry conditions

$$T_{kk_1,\,k_2k_3} = T_{k_1k,\,k_2k_3} = T_{kk_1,\,k_3k_2} = T^*_{k_2k_3,\,kk_1} \tag{2.4}$$

and k is either the whole real axis $-\infty < k < \infty$ or is $k = (p, q)$ that represents a real plane

$$-\infty < p < \infty \qquad -\infty < q < \infty.$$

For d=1 equation (2.1) is integrable, if

$$H = H^{(1)} + aH^{(2)}$$

Here a is an arbitrary constant.

Then:

$$\omega_k^{(1)} = k^2 \quad \omega_k^{(2)} = k^3$$

$$T^{(1)}_{k\,k_1\,k_2\,k_3} = \alpha$$

$$T^{(2)}_{k\,k_1\,k_2\,k_3} = \frac{3\alpha}{4}(k + k_1 + k_2 + k_3)$$

Thus:

$$\begin{aligned} H^{(1)} &= \int k^2 |\Psi_k|^2 dk + \frac{\alpha}{2} \int \Psi_k^* \, \Psi_{k_1}^* \, \Psi_{k_2} \, \Psi_{k_3} \, \delta_{k+k_1-k_2-k_3} \, dk \, dk_1 \, dk_2 \, dk_3 \\ H^{(2)} &= \int k^3 |\Psi_k|^2 dk + \frac{3\alpha}{4} \int (k + k_1 + k_2 + k_3) \, \Psi_k^* \, \Psi_{k_1}^* \, \Psi_{k_2} \, \Psi_{k_3} \, \times \\ & \qquad \times \delta_{k+k_1-k_2-k_3} \, dk \, dk_1 \, dk_2 \, dk_3 \end{aligned} \tag{2.5}$$

For equation (2.1):

$$\omega(k) = k^2 + a\,k^3 \tag{2.6}$$

$$T(k\,k_1\,k_2\,k_3) = \alpha\left[1 + \frac{3a}{4}(k + k_1 + k_2 + k_3)\right] \tag{2.7}$$

After Fourier transformation, it takes form:

$$\frac{\partial \Psi}{\partial t} = -i\,\Psi_{xx} + a\,\Psi_{xxx} - i\,\alpha|\Psi|^2\Psi + 3\,a\,\alpha|\Psi|^2\,\Psi_x \tag{2.8}$$

If $a = 0$ and $\alpha = -1$, this is a focusing Nonlinear Schrodinger equation. If $a = 0$ and $\alpha = 1$, this is a defocusing Nonlinear Schrodinger equation.

For $d = 2$ equation (2.3) is integrable if $k = (p, q)$, $\omega(k) = p^2 - q^2$, and

$$T(k\,k_1\,k_2\,k_3) = \frac{\alpha}{4}\left\{\frac{(p_1-p_2)^2-(q_1-q_2)^2}{(p_1-p_2)^2+(q_1-q_2)^2} + \frac{(p_1-p_3)^2-(q_1-q_3)^2}{(p_1-p_3)^2+(q_1-q_3)^2}\right\} \tag{2.9}$$

The coupling coefficient T is not yet properly symmetrized. Actually, it can be replaced by

$$T_{k\,k_1\,k_2\,k_3} \longrightarrow \frac{1}{2}\left[T(kk_1, k_2k_3) + T(k_1k,\ k_2,\ k_3)\right]$$

After the Fourier transformation, equation (2.3) becomes the Davey-Stewarson equation

$$\begin{aligned}\frac{\partial \Psi}{\partial t} &= \hat{i}\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi + \alpha\,U\,\Psi \\ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)U &= \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right)|\Psi|^2\end{aligned} \tag{2.10}$$

Here α is an arbitrary constant. One can put $\alpha = \pm 1$.

2.

$$\begin{aligned} H &= H_2 + H_3 \qquad N = 1 \\ H_2 &= \int \omega(k)\, |\Psi_k|^2\, dk \\ H_3 &= \int V_{kk_1k_3} \left(\Psi_k^* \Psi_{k_1} \Psi_{k_2} + \Psi_k \Psi_{k_1}^* \Psi_{k_2}^*\right)\, \delta(k - k_1 - k_2)\, dk\, dk_1\, dk_2 \end{aligned} \tag{2.11}$$

Equation (2.1) now reads:

$$\begin{aligned} \frac{\partial \Psi_k}{\partial t} &= i\,\omega_k\, \Psi_k + i \int \{ V_{kk_1k_2} \Psi_k\, \Psi_{k_2}\, \delta_{k-k_1+k_2} + \\ &+ \; 2V_{k_1k,\,k_2}\, \Psi_{k_1}\, \Psi_{k_2}^*\, \delta_{k-k_1+k_2} \}\, dk_1\, dk_2 \end{aligned} \tag{2.12}$$

Integrable versions of equation (2.13) are well known in $d = 1$. In this case $k = p$, $0 < p < \infty$ and

$$N_{kk_1k_2} = (p\, p_1\, p_2)^{1/2} \tag{2.13}$$

For $\omega(k)$ one can choose:

$\omega(p) = p^3$	KdV equation
$\omega(p) = p^2$	Benjamen-Ono equation
$\omega(p) = p^2 \coth pa$	Intermediate wave equation

If $d = 2$, the K-space should be half-plane: $p > 0$, $-\infty < q < \infty$. Again, we have to assume that $V_{kk_1k_2}$ is given by equation (2.14). As for $\omega(k)$, it can be chosen by two essentially different ways:

$$1. \quad \omega(p, q) = p^3 + \frac{3q^2}{p} \tag{2.14}$$

$$2. \quad \omega(p, q) = p^3 - \frac{3q^2}{p} \tag{2.15}$$

By transformation

$$U = \int_0^\infty dp \int_{-\infty}^\infty dq \,\sqrt{p} \left(\Psi_{p,q} + \Psi^*_{-p,-q}\right) e^{i(px+qy)} dp\, dq$$

equation (2.13) can be derived to the KP-equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right) = \alpha \frac{\partial^2 u}{\partial y^2} \quad (\alpha = \pm 1) \tag{2.16}$$

3.

Let $N = 3$ and Hamiltonian H is:

$$\begin{aligned}
H &= H_2 + H_3 \\
H_2 &= \sum \int \omega_i(k)\, |\Psi_i(k)|^2\, dk \\
H_3 &= \int V_{kk_1k_2} \left[\Psi_1^*(k_1)\, \Psi(k_2)\, \Psi(k_3) + \Psi_1(k_1)\, \Psi^*(k_2)\, \Psi^*(k_3)\right]\, \delta_{k_1-k_2-k_3}\, dk_1\, dk_2\, dk_3
\end{aligned} \tag{2.17}$$

Equation (2.1) turns now to:

$$\begin{aligned}
\frac{\partial \Psi_1}{\partial t} &= i\,\omega_1(k)\Psi_1 + i \int V_{kk_1k_2}\, \Psi(k_1)\, \Psi(k_2)\, \delta_{k-k_1-k_2}\, dk_1\, dk_2 \\
\frac{\partial \Psi_2}{\partial t} &= i\,\omega_2(k)\Psi_2 + i \int V_{k_1,k,k_2}\, \Psi_1(k_1)\, \Psi_3^*(k_2)\, \delta_{k+k_1-k_2}\, dk_1\, dk_2 \\
\frac{\partial \Psi_3}{\partial t} &= i\,\omega_3(k)\Psi_3 + i \int V_{k_1,k,k_2}\, \Psi_1(k_1)\, \Psi_2^*(k_2)\, \delta_{k-k_1+k_2}\, dk_1\, dk_2
\end{aligned} \tag{2.18}$$

Equations (2.19) are known as three-wave equations. They are integrable in dimensions $d = 1, 2$ if $V_{kk_1k_2} = V = const$ and $\omega_i(k)$ are linear functions. Without loosing of generality, one can assume:

$$\omega_1(k) = 0 \quad \omega_2(k) = (\vec{A}\,\vec{k}) \quad \omega_3(k) = (\vec{B}\,\vec{k})$$

Here $\vec{A}$, $\vec{B}$ are two-dimensional vectors. If they are not collinear, one can make the change of variables and put

$$\omega_2(u) = p \quad \omega_3(u) = q$$

If A, B are collinear, properties of three-wave system depend on the sign of (AB). If $(AB) = -1$, we can put $\omega_2 = p$, $\omega_3 = -p$. If $(AB) = 1$, we can put $\omega_2 = ap$, $\omega_3 = p/a$, $a \neq 1$. The case $a = 1$ is degenerative, and the three-wave system can be solved without use of Inverse Scattering Transform.

3 Derivation of kinetic equation

Equation (2.17) is the KP-1 equation if $\alpha = 1$, and is KP-2 equation if $\alpha = -1$. Since this moment we assume that $u(x,y,t)$ at any given t is a representative of homogeneous random field and $< u^2 >= I_1(t) \neq 0$. It means that $\Psi(p,q)$ is a generalized function, such that

$$< \Psi(k)\, \Psi^*(k') >= N(k)\, \delta_{k-k'} \tag{3.1}$$

$$< \Psi(k_1)\, \Psi^*(k_2)\, \Psi^*(k_3) >= I(k_1,k_2,k_3)\, \delta_{k_1-k_2-k_3} \tag{3.2}$$

As for the fourth-order correlations, we will assume

$$\begin{aligned} < \Psi(k)\, \Psi^*(k_1)\, \Psi^*(k_2)\, \Psi^*(k_3) > &= 0 \\ < \Psi(k)\, \Psi(k_1)\, \Psi^*(k_2)\, \Psi^*(k_3) > &= N(k)\, N(k_1)\, [\delta_{k-k_3}\, \delta_{k_1-k_3} + \delta_{k-k_2}\, \delta_{k_1-k_3}] \end{aligned} \tag{3.3}$$

Truncation (3.3) makes possible to construct a closed system of equations for N_k, $I_{kk_1k_2}$. They are:

$$\begin{aligned} \frac{\partial N_k}{\partial t} &= 2\int V_{kk_1k_2}\, Im\, I_{k,k_1k_2}\, \delta_{k-k_1-k_2}\, dk_1\, dk_2 - \\ &\quad -4\int V_{k_1,k,k_2}\, Im\, I_{k_1,k,k_2}\, \delta_{k-k_1+k_2}\, dk_1\, dk_2 \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{\partial}{\partial t} I_{k\,k_1\,k_2} &= i(\omega_k - \omega_{k_1} - \omega_{k_2})\, I_{k\,k_1\,k_2} + \\ &\quad +2\, i\, V_{k\,k_1\,k_2}(N_{k_1}\, N_{k_2} - N_k\, N_{k_1} - N_k\, N_{k_2}) \end{aligned} \tag{3.5}$$

Equation (3.5) is linear and inhomogeneous. If we assume that

$$I_{kk_1k_2}\big|_{t=0} = I^0_{kk_1k_2} \quad N(k)\big|_{t=0} = N_0(k),$$

then

$$\begin{aligned} I_{kk_1k_2} &= 2\,i\,V_{kk_1k_2}\int_0^t e^{i\,\Delta_{kk_1k_2}(\tau-t)}\,R_{kk_1k_2}(\tau)\,d\tau + I^0_{kk_1k_2} \\ R_{kk_1k_2} &= N_{k_1}\,N_{k_2} - N_k\,N_{k_1} - N_k\,N_{k_2} \\ \Delta_{kk_1k_2} &= \omega_k - \omega_{k_1} - \omega_{k_2} \end{aligned} \tag{3.6}$$

Let $t \to \infty$. Then everything depends on the following fundamental question: can we find a real solution of equations

$$\Delta_{kk_1k_2} = \omega_k - \omega_{k_1} - \omega_{k_2} = 0, \qquad \vec{k} = \vec{k_1} + \vec{k_2} \quad ? \tag{3.7}$$

One can see that for KP-2, where $\omega_k = p^3 - 3q^2/p$, this is impossible.

Then, if $t \to \infty$, N_k tends to some asymptotic value

$$N_k \longrightarrow N^\infty(k),$$

where

$$I^\infty_{kk_1k_2} \to -\frac{2\,V_{kk_1k_2}\,[N^\infty(k_1)\,N^\infty(k_2) - N^\infty(k)\,N^\infty(k_1) - N^\infty(k)\,N^\infty(k_2)]}{\omega(k) - \omega(k_1) - \omega(k_2)} \tag{3.8}$$

Notice, that $I^\infty_{kk_1k_2}$ is real. As for $N^\infty(k)$, we can make a conjecture that by a proper choice of $N_0(k)$, function $N^\infty(k)$ can become an arbitrary positive function on k.

4 Kinetic equation for KP-1 equation

The small amplitude waves in KP-1 equation are described by the standard 3-wave kinetic equation:

$$
\begin{aligned}
\frac{\partial N_k}{\partial t} &= 4\pi \Big\{ \int |V_{kk_1k_2}|^2 \, \delta_{k-k_1-k_2} \, \delta_{\omega_k-\omega_{k_1}-\omega_{k_2}} \, (N_{k_1}N_{k_2} - N_kN_{k_1} - N_kN_{k_2}) dk_1 \, dk_2 + \\
&+ \; 2 \; \int |V_{k_1,\,k,\,k_2}|^2 \, \delta_{k-k_1+k_2} \, \delta_{\omega_k-\omega_{k_1}+\omega_{k_2}} \; (N_{k-1}N_{k_2} - N_kN_{k_2} + N_kN_{k_1}) \, dk_1 dk_2 \Big\} = \\
&= Snl
\end{aligned}
\tag{4.1}
$$

However, this equation has some peculiar features that makes it completely different from similar equations in genetic nonintegrable systems. To trace these peculiarities, we should notice that the dispersion relation

$$
\omega(p,q) = p^3 + \frac{3q^2}{p}
$$

can be presented in the following parametric form:

$$
\begin{aligned}
p &= \xi - \eta \qquad \eta < \xi \\
q &= \xi^2 - \eta^2 \\
\omega &= 4(\xi^3 - \eta^3)
\end{aligned}
\tag{4.2}
$$

In variables ξ, η the resonant conditions

$$
\begin{aligned}
k &= k_1 + k_2 \\
\omega_k &= \omega_{k_1} + \omega_{k_2}
\end{aligned}
$$

have the following form:

$$\begin{aligned} \xi_1 - \eta_1 + \xi_2 - \eta_2 &= \xi - \eta \\ \xi_1^2 - \eta_1^2 + \xi_2^2 - \eta_2^2 &= \xi^2 - \eta^2 \\ \xi_1^3 - \eta_1^3 + \xi_2^3 - \eta_2^3 &= \xi^3 - \eta^3 \end{aligned} \tag{4.3}$$

Equations (4.3) have nontrivial solutions:

$$\begin{array}{lll} \xi_1 = \eta_2 & \xi_2 = \xi & \eta_1 = \eta \\ \xi_2 = \eta_1 & \xi_1 = \eta & \eta_2 = \eta \end{array} \tag{4.4}$$

In these variables equation (4.1) reads:

$$\begin{aligned} &\frac{\partial}{\partial t} N(\xi,\eta) = Snl = \\ &\frac{\pi}{3}\Big\{\int_\eta^\xi (\xi-\lambda)(\lambda-\eta)\,[N(\xi,\lambda)\,N(\lambda,\eta) - N(\xi,\eta)\,N(\xi,\lambda) - N(\xi,\eta)\,N(\lambda,\eta)]\,d\lambda + \\ &+ \int_{-\infty}^\eta (\eta-\lambda)(\xi-\lambda)\,[N(\xi,\lambda)\,N(\eta,\lambda) + N(\xi,\eta)\,N(\xi,\lambda) - N(\xi,\eta)\,N(\lambda,\eta)] + \\ &+ \int_\xi^\infty (\lambda-\eta)(\lambda-\xi)\,[N(\lambda,\xi)\,N(\lambda,\eta) + N(\xi,\eta)\,N(\lambda,\eta) - N(\xi,\eta)\,N(\lambda,\xi)]\,d\lambda\Big\} \end{aligned} \tag{4.5}$$

Equation (4.5) has infinite number of motion constants I_n

$$\frac{dI_n}{dt} = 0, \qquad I_n = \int_{-\infty}^\infty d\xi \int_\infty^\xi (\xi^n - \eta^n)(\xi-\eta)\,N(\xi,\eta)\,d\eta \tag{4.6}$$

Stationary equation

$$Snl = 0 \tag{4.7}$$

has infinite amount of exact solutions.

One can check that the function

$$N(\xi, \eta) = \frac{T}{f(\xi) - f(\eta)} \tag{4.8}$$

where T is a constant, satisfies equation (4.7). Solution (4.8) has a clear physical meaning: KP-1 equation is a member of a certain hierarchy of integrable equations. The linear part of each equation is:

$$\frac{\partial \Psi_k}{\partial t} = i\,\omega(k)\,\Psi_k + \cdots$$

Dispersion law $\omega(k) = \omega(p, q)$ can be presented in parametric form as follow:

$$\begin{aligned} p &= \xi - \eta \\ q &= \xi^2 - \eta^2 \\ f(p, q) &= f(\xi) - f(\eta) \end{aligned} \tag{4.9}$$

Solution (4.9) is the Rayley-Jeans solution corresponding to the dispersion relation (4.10). It has singularity on the diagonal $\xi = \eta$; on this diagonal $p = 0, q = 0$. This solution has no other singularities if $f(\xi)$ is a monotonically growing function on the axis $-\infty < \xi < \infty$, and represents a thermodynamic-type solution. As a rule, kinetic equation for waves has also Kolmogorov-type solutions, describing redistribution of energy along the spectrum. Solutions of this type for equation (4.6) are not found yet.

Higher members of the KP-1 hierarchy also have reasonable three-wave kinetic equations. They have the same set of motion constant (4.6) and the same exact solutions (4.9) as equation (4.5).

Three-wave system (2.17) in the generic integrable case

$$\omega_1 = 0 \qquad \omega_2 = p \qquad \omega_3 = q \qquad V = 1$$

also admits the statistical description in terms of kinetic equation. Assuming that

$$< \Psi_i(k)\, \Psi_i^*(k') >= N_i(k)\, \delta(k - k'),$$

after some calculation we will end up with the following system of equations:

$$\begin{aligned}
&\frac{\partial N_1(k)}{\partial t} = \\
&4\pi \int \{N_2(k_1)\, N_3(k_2) - N_1(k)\, N_2(k_1) - N_1(k)\, N_3(k_2)\}\, \delta_{k-k_1-k_2}\delta(p_1 + q_2) dk_1 dk_2 \\
&\frac{\partial N_2(k)}{\partial t} = \\
&4\pi \int \{N_1(k_1)\, N_3(k_2) - N_2(k_1)\, N_1(k_1) - N_2(k_1)\, N_3(k_2)\}\, \delta_{k-k_1-k_2}\delta(p - q_2) dk_1 dk_2 \\
&\frac{\partial N_3(k)}{\partial t} = \\
&4\pi \int \{N_1(k_1)\, N_2(k_2) - N_3(k)\, N_1(k_1) - N_3(k)\, N_2(k_2)\}\, \delta_{k-k_1-k_2}\delta(q - p_1) dk_1 dk_2
\end{aligned} \tag{4.10}$$

As equation (4.5), equations (4.10) have infinite amount of exact thermodynamic solutions. In a given presentation we don't have enough time to discuss these solutions in details.

5 Absence of higher-order kinetic equations

In the previous chapter we have seen that the statistical properties of some integrable systems (KP-1, 3-wave equation) can be described by three-wave kinetic equation. If for some reason three-wave resonances are forbidden and we will try to construct high-order kinetic equation, we will inevitably fail. Again, let us start with examples.

Let us consider equation (2.2). Using a procedure, similar to described in Chapter 3, we easily can construct a closed system of equations for N_k and a forth-order cumulant, which can be defined as follow:

$$Im \; < \Psi^*_{k_1} \, \Psi^*_{k_2} \, \Psi_{k_3} \, \Psi_{k_4} > = I_{k_1\, k_2\, k_3\, k_4} \, \delta_{k_1+k_2-k_3-k_4} \tag{5.1}$$

Equation for $I_{k_1\, k_2\, k_3\, k_4}$ can be resolved by a standard way, and we will end up with a standard kinetic equation:

$$\begin{aligned} &\frac{\partial N(k)}{\partial t} = 4\,\pi \int |T_{k\, k_1\, k_2\, k_3}|^2 \, \delta(k + k_1 - k_2 - k_3) \, \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \times \\ &(N_{k_1}\, N_{k_2}\, N_{k_3} + N_k\, N_{k_2}\, N_{k_3} - N_k\, N_{k_1}\, N_{k_2} - N_k\, N_{k_1}\, N_{k_3}) \, dk_1\, dk_2\, dk_3 = \\ &= Snl \end{aligned} \tag{5.2}$$

Let us try to construct the kinetic equation for generalized NSLE (2.8). In this case

$$\omega_k = k^2 + a\, k^3$$

The resonant manifold

$$\omega_k + \omega_{k-1} = \omega_{k_2} + \omega_{k_3}, \qquad k + k_1 = k_2 + k_3 \tag{5.3}$$

can be reduced to one algebraic equation. Assuming that

$$k = P + p, \qquad k_1 = P - p, \qquad k_2 = P + q, \qquad k_3 = P - q \tag{5.4}$$

we find that (5.3) is equivalent to equation

$$(p^2 - q^2)(1 + 3\,a\,P) = 0. \tag{5.5}$$

For the case $q = \pm p$, we have trivial resonances:

$$q = p: \quad k_2 = k,\ k_1 = k_3 \qquad q = -p: \quad k_3 = k,\ k_1 = k_2 \tag{5.6}$$

Obviously, for them $Snl \equiv 0$. Nontrivial inelastic resonances take place if $1 + 3\,a\,P = 0$. However, by plugging (5.4) into (2.7) we find that

$$T(k,\ k_1,\ k_2,\ k_3) = \alpha(1 + 3\,a\,P) \equiv 0$$

A similar situation takes place for the Davey-Stewardson equation (2.10). Now

$$\omega(p,\ q) = p^2 - q^2$$

and the resonant manifold

$$\begin{aligned} \omega(p,\ q) + \omega(p_1,\ q_1) &= \omega(p_2,\ q_2) + \omega(p_3,\ q_3) \\ p + p_1 &= p_2 + p_3 \\ q + q_1 &= q_2 + q_3 \end{aligned} \tag{5.7}$$

can be reduced to one equation, if we put

$$
\begin{aligned}
&p = P + \xi_1, \quad p_1 = P - \xi_1, \quad p_2 = P + \xi_2, \quad p_3 = P - \xi_2 \\
&q = Q + \eta_1, \quad q_1 = Q - \eta_1, \quad q_2 = Q + \eta_2, \quad q_3 = Q - \eta_2
\end{aligned} \tag{5.8}
$$

By plugging (5.8) into (5.7), we derive the equation

$$
\xi_1^2 + \xi_2^2 - \eta_1^2 - \eta_2^2 = 0 \tag{5.9}
$$

Plugging (5.8) to (2.9), we find that

$$
T_{k\,k_1\,k_2\,k_3} \sim (\xi_1^2 + \xi_2^2 - \eta_1^2 - \eta_2^2)^2 = 0
$$

In this case trivial resonances are not separated from nontrivial. They form a connected manifold, where $T(kk_1k_2k_3) \simeq 0$.

As we know, for KP-2 equation the three-wave resonances are forbidden. Of course, four-wave resonances are allowed. One can perform a canonical transformation, excluding cubic nonlinearity in the Hamiltonian. The four-wave resonances are described by equation

$$
\begin{aligned}
p^2 - \frac{3\,q^2}{p} + p_1^2 - \frac{3q_1^2}{p_1} &= p_2^2 - \frac{3\,q_2^2}{p_2} + p_3^2 - \frac{3\,q_3^2}{p_3} \\
p + p_1 &= p_2 + p_3 \\
q + q_1 &= q_2 + q_3
\end{aligned} \tag{5.10}
$$

The expression for effective four-wave coupling coefficient $T(kk_1, k_2k_3)$ is pretty complicated. It can be found in the article []. Nevertheless, in the same article was directly demonstrated that $T(kk_1k_2k_3) \simeq 0$ on the manifold (5.10).

For higher order processes the situation is as bad as for four-wave interaction. In article [] was demonstrated that the amplitude of six-order processes on the resonant manifold is identically zero. For both KdV and Benjamin-Ono equations, the first nontrivial process is five-wave interaction. It is easy to prove that this amplitude is identically zero. This result will be published soon.

6 Turbulence in strong integrable systems

Let us return to NSLE and treat it as a typical representative of strongly integrable system. We propose that this equation has infinite number of statistically stationary states parameterized by an arbitrary positive function of one variable $N(k)$. The condition

$$\frac{dN}{dt} = 0$$

makes possible, at least in principle, to find all higher order correlation functions and reconstruct the invariant measure in the functional space. The stationary state is spatially uniform. It means that one can introduce a set of constants:

$$\begin{aligned} I_1 &= \lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} |\Psi|^2\, dx \\ I_2 &= \lim_{L\to\infty} \frac{1}{L}\int_{-L/2}^{L/2} \left\{|\Psi_x|^2 - \frac{1}{4}|\Psi|^4\right\} dx \\ I_3 &= \ldots \end{aligned} \tag{6.1}$$

These constants are densities of commuting motion integrals. Existence of invariant spectrum $N(k)$ presumes existence of invariant measure. One can guess that this measure is nothing but the Gibb's measure:

$$\rho[\Psi] = \frac{1}{z}\, e^{-\sum_{i=1}^{\infty} \mu_i\, I_i} \tag{6.2}$$

Here μ_i are "chemical potentials", corresponding to given motion constants, and z is the statistical sum given by functional integral

$$z = \int e^{-\sum_{i=1}^{\infty} \mu_i I_i} \, d\Psi(x) \, d\Psi^*(x) \tag{6.3}$$

Each stationary state is characterized by the probability distribution function

$$\rho(\xi) = \rho(|\Psi|^2), \qquad \int_0^\infty \rho(\xi) \, d\xi \tag{6.4}$$

One can guess that any stationary state is completely defined by the set of constants $I_1, I_2, \ldots$. As far as NLSE is the scale invariant equation, one can put without violation of generality that $I_1 = 1$. Then the basic physical properties of the stationary state are defined in large degree by the value of I_2. If $I_2 \to \infty$, this is a state close to superposition of weakly interacting, almost linear waves. On the contrary, if $I_2 \to -\infty$, this state is the solitonic gas superposition of well separated weakly interacting solitons. The both cases can be studied efficiently but they need completely different treatment. In spite of illusory simplicity of this theory, some important questions are not yet answered.

The most important one is the question about modulational instability. One of the stationary states in the Bose-condensate

$$N(k) = \delta(k)$$

In the defocusing NLSE the condensate is stable, but in the focusing case it is unstable. Development of this instability generates something intermediate between weak turbulence and solitonic gas. The theory of condensate instability is pure dynamical and simple.

Much more difficult is the question of stability of "broaden" condensate

$$N(k) = \frac{1}{\pi} \frac{\gamma}{(k^2 + \gamma^2)} \tag{6.5}$$

Again we assume that $< N(k) >= 1$. One can study stability of this distribution in framework of the mean-field approximation. This approximation can be used for study of long-scale perturbations with a characteristic wave number much less than γ. In spite of the fact that the homogeneous kinetic equation does not exist, the inhomogeneous kinetic equation makes sense. If we suppose that N is also a function of "slow" variables x, t, we can write the following "Vlasov-type" equation

$$\frac{\partial N}{\partial t} + k \frac{\partial N}{\partial x} - \frac{\partial N}{\partial k} \frac{\partial n}{\partial x} = 0, \qquad n = - \int_{-\infty}^{\infty} N(k)\, dk \tag{6.6}$$

Now one can assume

$$N = N(k) + \delta\, N\, e^{-i\omega\, t + i\, \rho\, x}$$

and end up with characteristic equation

$$\int_{-\infty}^{\infty} \frac{1}{s - k} \frac{\partial N}{\partial k}\, dk = -1 \tag{6.7}$$

Here $s = \omega/p$.

By plugging (6.5) to (6.7) and calculating the integral, one finds easily

$$s = -i(\gamma - 1) \tag{6.8}$$

In other words, distribution is stable if $\gamma > 1$, and unstable if $\gamma < 1$. This consideration is nice but has a weak point. According to (6.8)

$$\omega = -i(\gamma - 1)p \tag{6.9}$$

Thus, $Im\omega \to \infty$ as $p \to \infty$.

It is clear that in reality

$$Im\omega = -(1-\gamma)p + qp^2 + \cdots$$

$q > 0$ is some positive constant depending on γ. Determination of this constant is a question of theoretical and practical importance. Apparently it cannot be done in framework of the mean-field approximation.

The second fundamental question is the intermittency or structure of higher momentum

$$I_n(y) = |\Psi(x+y) - \Psi(x)|^{2n}$$

This question is interesting when

$$I_2 < 0,\ |I_2| >> 1$$

In this case the stationary case is a solitonic gas defined by the distribution function on soliton amplitudes. The higher moments, as far as the PDF (6.4) should be directly expressed in terms of distribution function for solitons. Theory of solitonic gas is a very interesting subject deserving a special consideration.