Breakdown of the Equivalence between Gravitational Mass and Energy for a Composite Quantum Body

International Conference "Landau Days"
(Chernogolovka, June 24, 2013)

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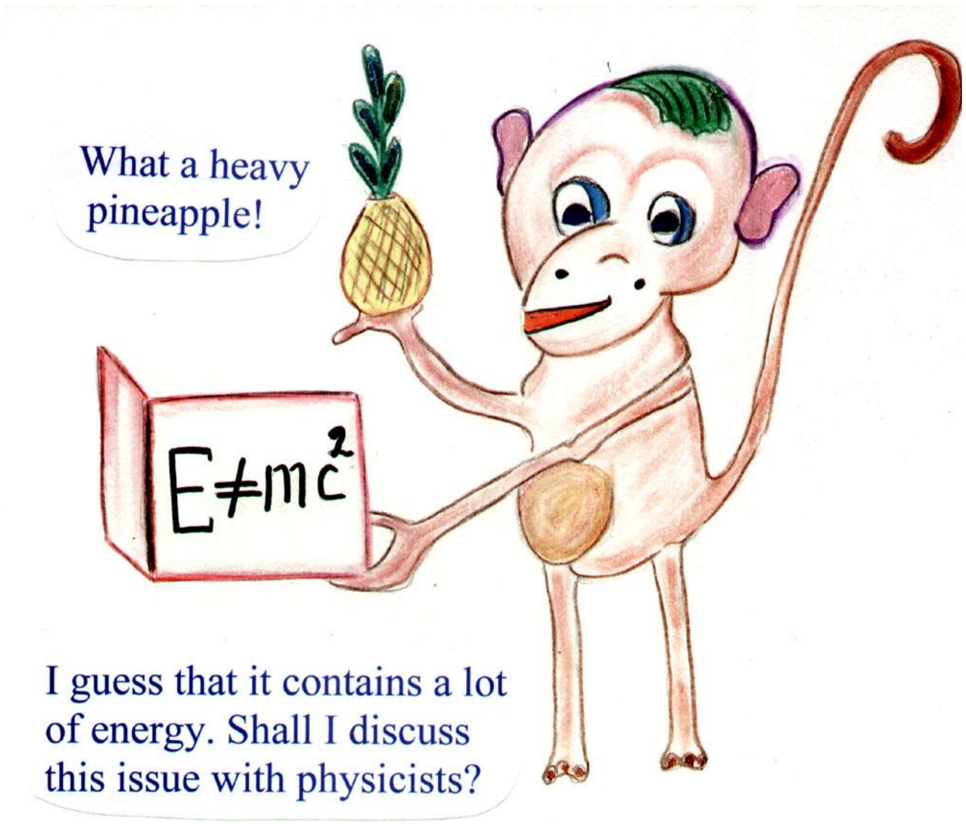
NSF grant DMR-1104512

The most ancient question

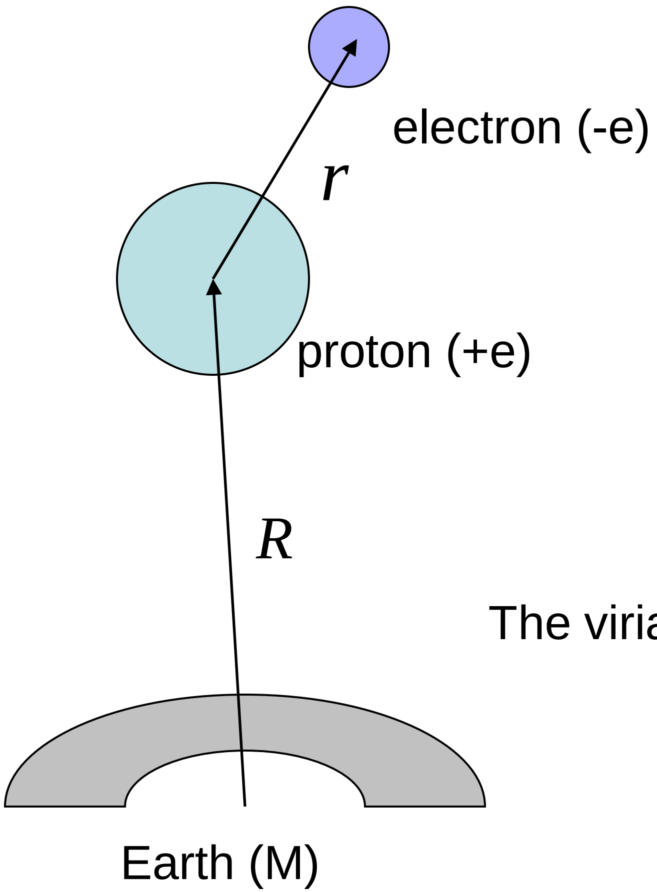


Drawing of Natalia Lebed

Five aspects of our work

- 1) One more step towards the "Theory of Everything".
- 2) Confirmation of the Einstein equation, $E = m_g c^2$, with accuracy $\propto 10^{-18}$.
- 3) Breakdown of the Einstein equation with probability $\propto 10^{-18}$.
- 4) Suggestion of the experiment in satellite or spacecraft on the Earth's orbit to detect such seldom events, where $E \neq m_g c^2$.
- 5) Suggestion of the first experiment, where some combination of quantum mechanics and general relativity is tested.

Gravitational mass of a composite body in classical physics



$$V(R) = \left(m_p + m_e + \frac{3K + 2U}{c^2} \right) \phi(R)$$

$$\langle m_e^g \rangle_t = m_e + \left\langle \frac{K + U}{c^2} \right\rangle_t + \left\langle \frac{2K + U}{c^2} \right\rangle_t = m_e + \frac{E}{c^2}$$

The virial theorem: $\langle 2K \rangle_t = \langle \vec{r} \vec{\nabla} U \rangle_t = -\langle U \rangle_t$

$$\langle m_e^g \rangle_t = m_e + E / c^2$$

K. Nordtvedt, Class. Quantum Grav., v. 11, p. A119 (1994)
S. Carlip, Am. J. Phys., v. 66, p. 409 (1998)

Kinetic energy term's contribution

$$S_{kin} = -mc^2 \int_A^B d\tau \qquad d\tau = -ds$$

$$S_{kin} = -mc^2 \int_A^B dt \left[\left(1 + \frac{2\varphi}{c^2}\right) - \frac{1}{c^2}\left(1 - \frac{2\varphi}{c^2}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right] \right]^{1/2}$$

$$S_{kin} = \int_A^B L dt \qquad L = -mc^2 \left[\left(1 + \frac{2\varphi}{c^2}\right) - \frac{1}{c^2}\left(1 - \frac{2\varphi}{c^2}\right) v^2 \right]^{1/2}$$

$$L = -mc^2 + \frac{1}{2} m v^2 - \varphi m - \varphi \frac{3}{2} m \frac{v^2}{c^2}$$

$$m \rightarrow m + \frac{3}{2} m \frac{v^2}{c^2}$$

Electromagnetic Field (potential energy)

$$S_{em} = \frac{1}{8\pi c}\int d^4x\left(E^2 - B^2\right)$$

$$S_{em} = -\frac{1}{16\pi c}\int dx^4\sqrt{-g}\,F_{\mu\nu}F_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta} \qquad g = \det g_{\alpha\beta}$$

$$S_{em} = \frac{1}{8\pi c}\int dx^4\left[\left(1-\frac{2\varphi}{c^2}\right)E^2 - \left(1+\frac{2\varphi}{c^2}\right)B^2\right]$$

$$\varepsilon = \mu = 1-\frac{2\varphi}{c^2} \qquad -\frac{e^2}{r} \to -\frac{e^2}{\varepsilon r} \approx -\frac{e^2}{r}\left(1+\frac{2\varphi}{c^2}\right)$$

$$m \to m - 2\frac{e^2}{rc^2}$$

Weak external gravitational field; No tidal effects

$$ds^2 = -\left(1 + 2\frac{\phi}{c^2}\right)(cdt)^2 + \left(1 - 2\frac{\phi}{c^2}\right)(dx^2 + dy^2 + dz^2),$$

$$\phi = -\frac{GM}{R}, \qquad (2)$$

$$|\varphi / c^2| \propto 10^{-9}, r_B / R_0 \propto 10^{-17}$$

Then in the local proper spacetime coordinates,

$$x' = \left(1 - \frac{\phi}{c^2}\right)x,\ y' = \left(1 - \frac{\phi}{c^2}\right)y,$$

$$z' = \left(1 - \frac{\phi}{c^2}\right)z,\ t' = \left(1 + \frac{\phi}{c^2}\right)t, \qquad (3)$$

the classical Lagrangian and action of an electron in a hydrogen atom have the following standard forms:

$$L' = -m_e c^2 + \frac{1}{2}m_e(\mathbf{v}')^2 + \frac{e^2}{r'}, \quad S' = \int L' dt', \qquad (4)$$

Equivalence between passive gravitational mass, averaged over time, and energy in classical physics

$$L = -m_e c^2 + \frac{1}{2} m_e \mathbf{v}^2 + \frac{e^2}{r} - m_e \phi - \left(3 m_e \frac{\mathbf{v}^2}{2} - 2 \frac{e^2}{r} \right) \frac{\phi}{c^2}. \quad (5)$$

Let us calculate the Hamiltonian, corresponding to the Lagrangian (5), by means of a standard procedure, $H(\mathbf{p}, \mathbf{r}) = \mathbf{p}\mathbf{v} - L(\mathbf{v}, \mathbf{r})$, where $\mathbf{p} = \partial L(\mathbf{v}, \mathbf{r}) / \partial \mathbf{v}$. As a result, we obtain:

$$H = m_e c^2 + \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} + m_e \phi + \left(3 \frac{\mathbf{p}^2}{2m_e} - 2 \frac{e^2}{r} \right) \frac{\phi}{c^2}, \quad (6)$$

$$< m_e^g \phi >_t = m_e \phi + \left\langle \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} \right\rangle_t \frac{\phi}{c^2} + \left\langle 2 \frac{\mathbf{p}^2}{2m_e} - \frac{e^2}{r} \right\rangle_t \frac{\phi}{c^2}$$

$$= \left(m_e + \frac{E}{c^2} \right) \phi \,, \quad (7)$$

K. Nordtvedt, Class. Quantum Grav., v. 11, p. A119 (1994)
S. Carlip, Am. J. Phys., v. 66, p. 409 (1998)

Classical virial theorem

$$S = \vec{p}\vec{r}$$

$$\left\langle \frac{dS}{dt} \right\rangle_t = \lim_{T\to\infty} \int_0^T \frac{dS}{dt} dt = \lim_{T\to\infty} \frac{S(T) - S(0)}{T} = 0$$

$$\frac{dS}{dt} = \vec{p}\dot{\vec{r}} + \dot{\vec{p}}\vec{r} = mv^2 + \vec{f}\vec{r}$$

$$\vec{p}\dot{\vec{r}} = mv^2$$

$$\dot{\vec{p}} = \vec{f}$$

$$\langle 2K \rangle_t = -\langle \vec{f}\vec{r} \rangle_t = \langle \vec{r}\vec{\nabla} U \rangle_t = \left\langle \frac{e^2}{r} \right\rangle_t = -\langle U(r) \rangle_t$$

Equivalence between the expectation values of gravitational mass and energy in quantum case

cal momentum, p. It is convenient to write the quantized Hamiltonian in the following form:

$$\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi \, , \qquad (8)$$

where we omit term $m_e c^2$ and introduce weight operator of an electron in a weak gravitational field,

$$\hat{m}_e^g \phi = m_e \phi + \left(\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) \frac{\phi}{c^2} + \left(2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r} \right) \frac{\phi}{c^2} \, . \qquad (9)$$

$$< \hat{m}_e^g \phi > = m_e \phi + \frac{E_1}{c^2}\phi + \left\langle 2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} \right\rangle \frac{\phi}{c^2} = \left(m_e + \frac{E_1}{c^2} \right) \phi, \qquad (10)$$

Quantum Virial Theorem: $\langle 2K \rangle = \langle \vec{r}\vec{\nabla} U \rangle = \left\langle \frac{e^2}{r} \right\rangle$ $\quad \langle \hat{m}_e^g \rangle = m_e + E_1 / c^2 \quad 10^{-18}$

A.G. Lebed, arXiv: 1111.5365v1 [gr-qc] (2011); ibid. 1205.3134v1 [gr-qc] (2012); A.G. Lebed, Talk at Marcel Grossmann Meeting-13 (arXiv: 1208.5756v1 [gr-qc]), to be published; A.G. Lebed, Phys. Rev. Lett., submitted (2013)

Quantum Virial theorem for stationary quantum states

$$\hat{S} = \frac{1}{2}\left(\vec{r}\hat{\vec{p}} + \hat{\vec{p}}\vec{r}\right)$$

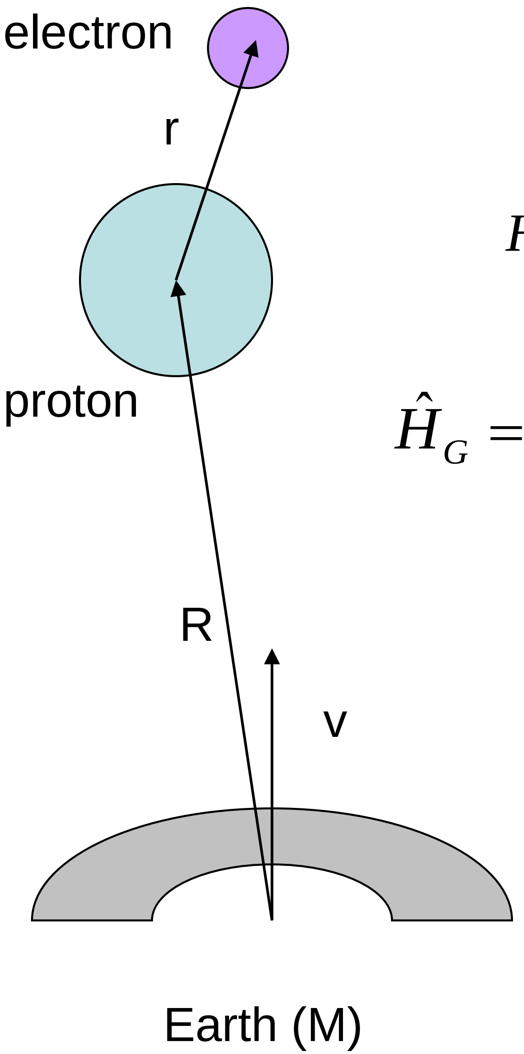
$$\frac{d}{dt}\left\langle \hat{S} \right\rangle = \left\langle \frac{i}{\hbar}\left[\hat{H}, \hat{S}\right] \right\rangle + \left\langle \frac{\partial S}{\partial t} \right\rangle$$

$$\left\langle \frac{i}{\hbar}\left[\hat{H}, \hat{S}\right] \right\rangle = 0$$

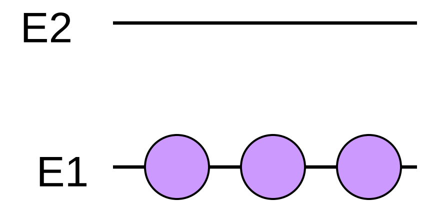
$$\hat{H} = \frac{\hat{\vec{p}}^{\,2}}{2m} + V(r)$$

$$\left\langle 2\hat{K} \right\rangle = \left\langle \vec{r}\vec{\nabla}U \right\rangle = \left\langle \frac{e^2}{r} \right\rangle$$

Gedanken experiment-1



$$\psi_1(r,t) = \psi_1(r)\exp\left(-iE_1 t/\hbar\right)$$



$$\hat{H}_0 = \frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r} \qquad \hat{H} = \hat{H}_0 + \hat{H}_G$$

$$\hat{H}_G = \varphi \exp(\lambda t)\left[m_e + \left(\frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r} \right)/c^2 + \left(2\frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r} \right)/c^2 \right]$$

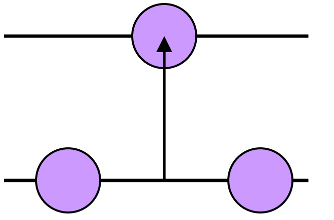
$$\lambda \to 0$$

$$\psi(r,t) = \sum_{n=1}^{\infty} a_n(t)\psi_n(r)\exp\left(-iE_n t/\hbar\right)$$

$$a_1(t) = \exp\left[-i\varphi\left(m_e + E_1/c^2\right)t/\hbar\right]$$

Mass quantization and the corresponding probabilities

$$a_n(0) = -\frac{\varphi}{c^2}\frac{V_{n,1}}{E_n - E_1} \quad for \quad n \neq 1$$



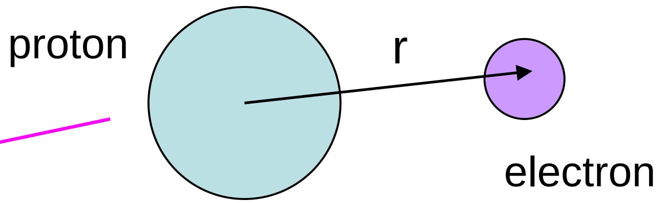
$$V_{n,1} = \int \psi_n^*(r)\hat{V}(\vec{r})\psi_1(r)d^3\vec{r} \qquad \hat{V}(\vec{r}) = 2\frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r}$$

$$m_e^g(n) = m_e + E_n / c^2$$

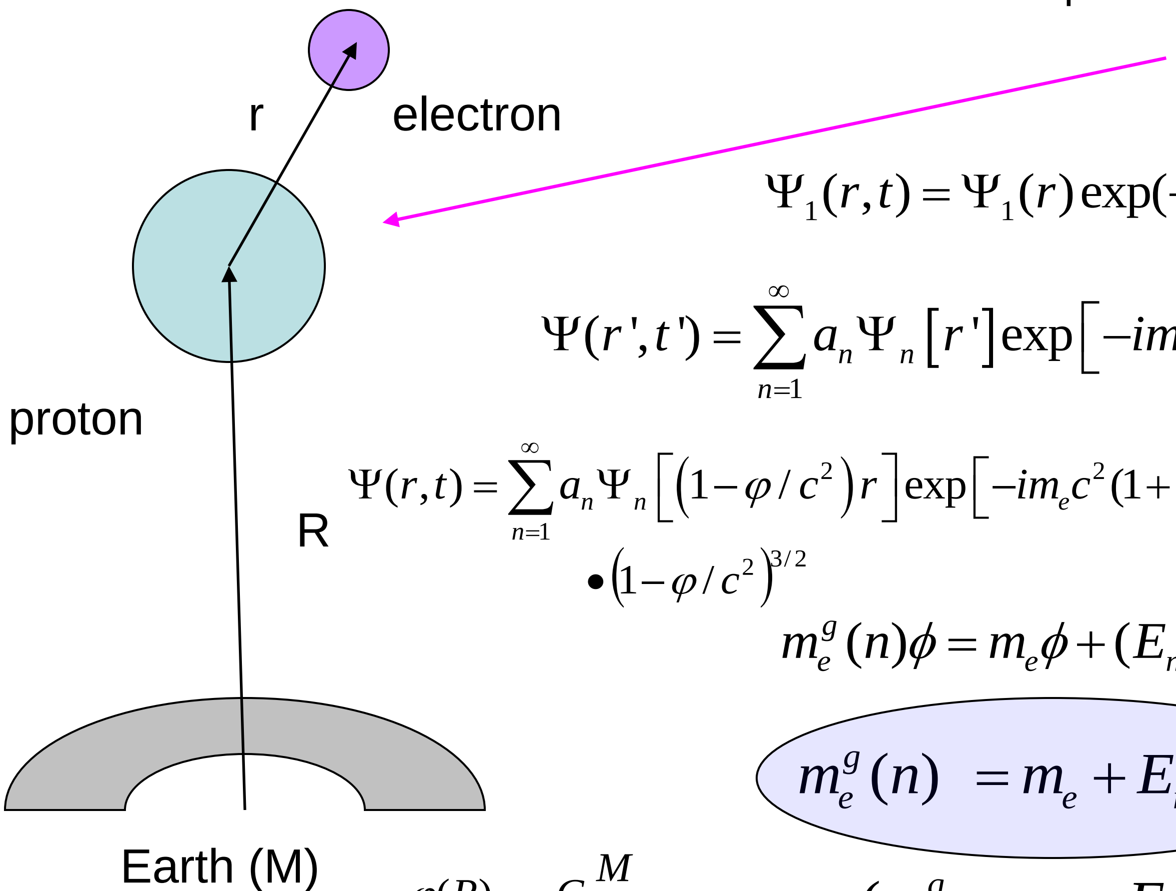
$$P_n = |a_n(0)|^2 = \left(\frac{\varphi}{c^2}\right)^2 \left(\frac{V_{n,1}}{E_n - E_1}\right)^2 \quad for \quad n \neq 1$$

A.G. Lebed, arXiv: 1111.5365v1 [gr-qc] (2011); ibid. 1205.3134v1 [gr-qc] (2012); A.G. Lebed, Talk at Marcel Grossmann Meeting-13 (arXiv: 1208.5756v1 [gr-qc]), to be published; A.G. Lebed, Phys. Rev. Lett., submitted (2013).

Gedanken experiment-2 and mass quantization



$$\Psi_1(r,t) = \Psi_1(r)\exp(-im_e c^2 t/\hbar)\exp(-iE_1 t/\hbar)$$



$$\Psi(r',t') = \sum_{n=1}^{\infty} a_n \Psi_n[r']\exp\left[-im_e c^2 t'/\hbar\right]\exp\left[-iE_n t'/\hbar\right]$$

$$\Psi(r,t) = \sum_{n=1}^{\infty} a_n \Psi_n\left[\left(1-\varphi/c^2\right)r\right]\exp\left[-im_e c^2(1+\varphi/c^2)t/\hbar\right]\exp\left[-iE_n(1+\varphi/c^2)t/\hbar\right]$$

$$\bullet\left(1-\varphi/c^2\right)^{3/2}$$

$$m_e^g(n)\phi = m_e\phi + (E_n/c^2)\phi$$

$$\varphi(R) = -G\frac{M}{R}$$

$$m_e^g(n) = m_e + E_n/c^2$$ instead of

$$(m_e^g = m_e + E_1/c^2)$$

A.G. Lebed, arXiv: 1111.5365v1 [gr-qc] (2011); ibid. 1205.3134v1 [gr-qc] (2012); Marcel Grossmann Meeting–13 (arXiv: 1208.5756v1 [gr-qc] (2012)), to be published.

Eigenfunctions of in a gravitational field

$$\Psi_1(r,t) = \Psi_1(r)\exp(-im_e c^2 t/\hbar)\exp(-iE_1 t/\hbar)$$

$$\hat{H} = mc^2 + \frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r} + m_e\varphi + \left(3\frac{\hat{\vec{p}}^2}{2m_e} - 2\frac{e^2}{r}\right)\varphi/c^2$$

$$\left[mc^2 + \frac{\hat{\vec{p}}^2}{2m_e} - \frac{e^2}{r} + m_e\varphi + \left(3\frac{\hat{\vec{p}}^2}{2m_e} - 2\frac{e^2}{r}\right)\varphi/c^2\right]\psi_n\left[\left(1-\frac{\varphi}{c^2}\right)r\right] =$$

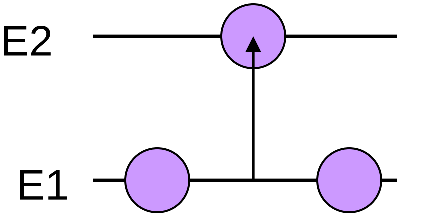
$$\left[mc^2\left(1+\frac{\phi}{c^2}\right) + E_n\left(1+\frac{\phi}{c^2}\right)\right]\psi_n\left[\left(1-\frac{\varphi}{c^2}\right)r\right]$$

$$\psi_n\left[\left(1-\frac{\varphi}{c^2}\right)r\right]$$

The corresponding probabilities

$$P_n = |a_n|^2, \;\; a_n = \int \Psi_1^*(r)\Psi_n[(1-\phi/c^2)r]d^3\mathbf{r}$$

$$= -(\phi/c^2)\int \Psi_1^*(r)r\Psi_n'(r)d^3\mathbf{r}, \quad n \neq 1. \qquad (13)$$



Taking into account that the Hamiltonian is the Hermitian operator, it is possible to show that

$$\int \Psi_1^*(r)r\Psi_n'(r)d^3\mathbf{r} = \frac{V_{n,1}}{\hbar\omega_{n,1}}, \quad \hbar\omega_{n,1} = E_n - E_1, \qquad (14)$$

where

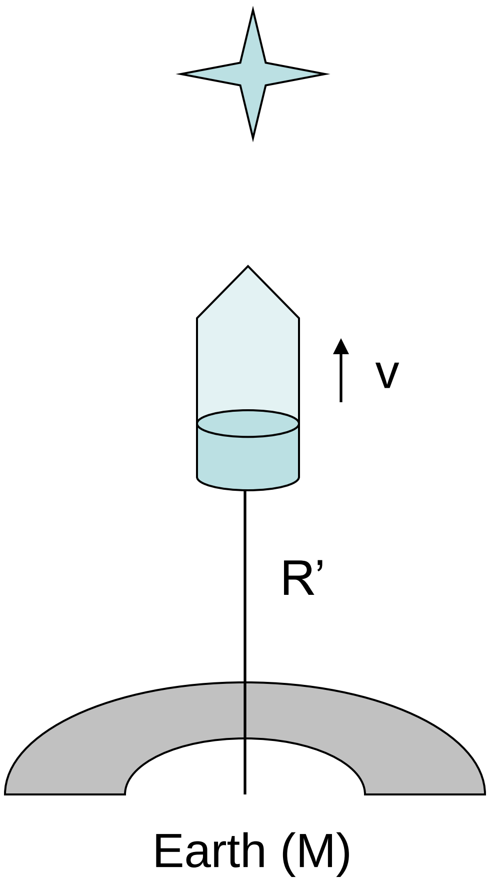
$$V_{n,1} = \int \Psi_1^*(r)\hat{V}(\mathbf{r})\Psi_n(r)d^3\mathbf{r}, \quad \hat{V}(\mathbf{r}) = 2\frac{\hat{\mathbf{p}}^2}{2m_e} - \frac{e^2}{r}. \qquad (15)$$

Let us discuss Eqs.(12)-(15). Note that they directly demonstrate that there is a finite probability,

$$P_n = |a_n|^2 = \left(\frac{\phi}{c^2}\right)^2 \left(\frac{V_{n,1}}{E_n - E_1}\right)^2, \; n \neq 1, \qquad (16)$$

$$(\varphi/c^2)^2 \propto 10^{-18}$$

Realistic experiment on the Earth's orbit by using spacecraft or satellite



$$\tilde{\Psi}_1(r,t) = \Psi_1[(1-\phi'/c^2)r]\exp[-iE_1(1+\phi'/c^2)t/\hbar] \;, \tag{18}$$

$$\tilde{\Psi}(r,t) = \sum_{n=1}^{\infty} \tilde{a}_n(t)\Psi_n[(1-\phi'/c^2)r]\exp[-iE_n(1+\phi'/c^2)t/\hbar], \tag{19}$$

$$\hat{U}(\mathbf{r},t) = \frac{\phi(R'+vt)-\phi(R')}{c^2}\left(3\frac{\hat{\mathbf{p}}^2}{2m_e} - 2\frac{e^2}{r}\right). \tag{20}$$

$$\tilde{a}_n(t) = \frac{\phi(R')-\phi(R'+vt)}{c^2}\frac{V_{n,1}}{\hbar\omega_{n,1}}\exp(i\omega_{n,1}) \;,\; n \neq 1 \;,$$

A.G. Lebed, arXiv: 1111.5365v1 [gr-qc] (2011); ibid. 1205.3134v1 [gr-qc] (2012);
A.G. Lebed, Talk at Marcel Grossmann Meeting-13 (arXiv: 1208.5756v1 [gr-qc]), to be published; A.G. Lebed, Phys. Rev. Lett., submitted (2013)

The corresponding probabilities

$$\tilde{P}_n(t) = \left(\frac{V_{n,1}}{\hbar\omega_{n,1}}\right)^2 \frac{[\phi(R' + vt) - \phi(R')]^2}{c^4}, n \neq 1. \quad (22)$$

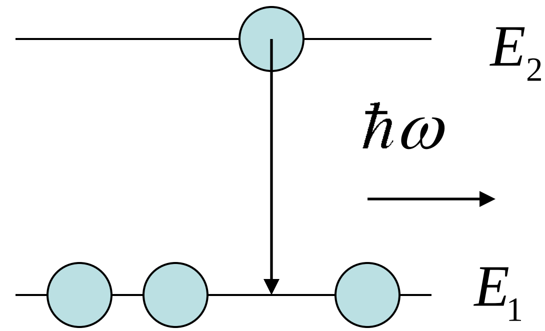
$$\tilde{P}_n = \left(\frac{V_{n,1}}{E_n - E_1}\right)^2 \frac{\phi^2(R')}{c^4} \simeq 0.49 \times 10^{-18} \left(\frac{V_{n,1}}{E_n - E_1}\right)^2 \quad (23)$$

For 1000 moles

$$N(n \to 1) = 2.95 \times 10^8 \left(\frac{V_{n,1}}{E_n - E_1}\right)^2,$$

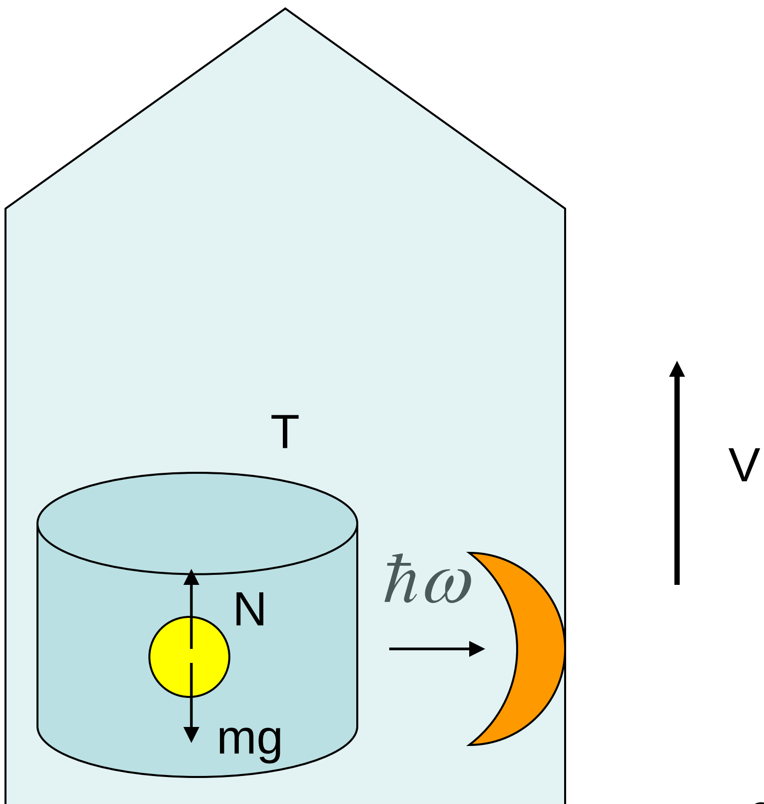
$$N(2 \to 1) = 0.9 \times 10^8, \quad (24)$$

Inside the spacecraft



$$\hbar\omega \approx 10ev \approx 120{,}000K$$

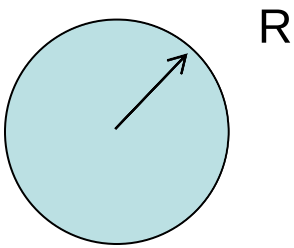
$$\frac{\Delta\omega}{\omega} \propto \frac{1}{\alpha^2}\frac{\varphi}{c^2}\frac{r_B}{R_0} \propto 10^{-22}$$



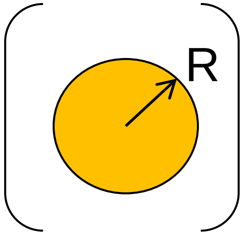
$$\exp\left(-\frac{\hbar\omega}{k_b T_R}\right) \approx \exp(-400) \approx 10^{-175}$$

$$\exp\left(-\frac{\hbar\omega}{k_b T_B}\right) \approx \exp(-6000) \approx 10^{-2500}$$

More details of the experiment

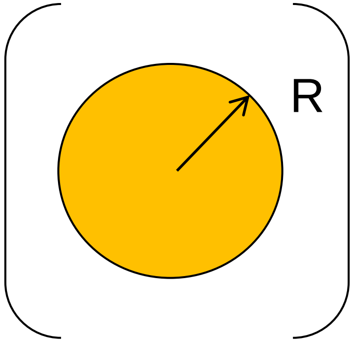


Avogadro law: for 1 mole (T = 0 C, P = 1 bar), V = 22.4 l,
R = 17 cm



$$N = 10^5$$

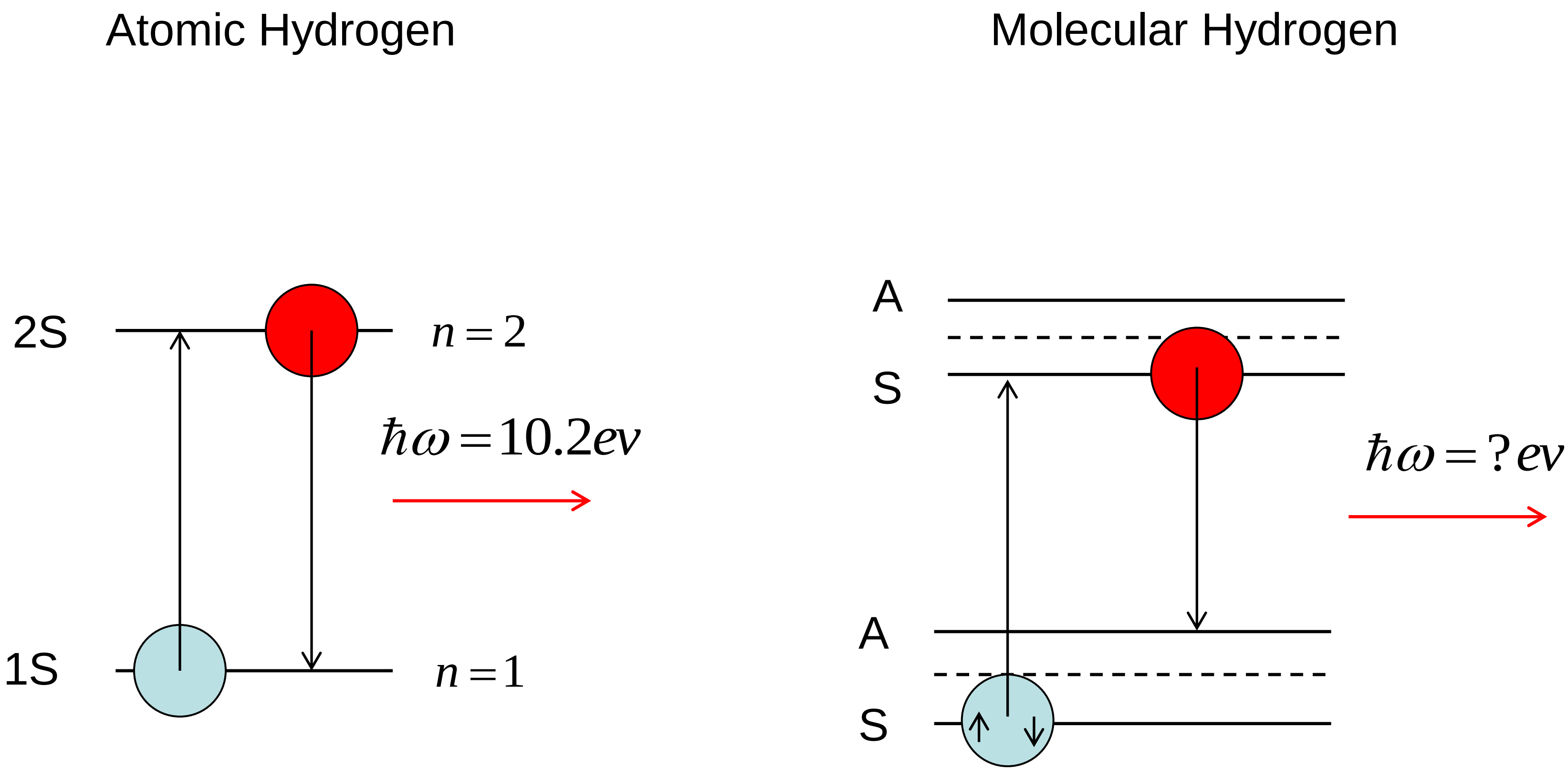
1 mole of pressured H_2: (T = 0 C, P = 1 kbar), R = 1.7 cm



$$N = 10^8$$

1 kmole of pressured H_2: (T = 0 C, P = 1 kbar), R = 17 cm

Molecular Hydrogen



Summary

- 1) The Einstein equation, $E = m^g c^2$, is established for the expectation values of passive gravitational mass and energy;
- 2) It is shown to be broken with probability $P \propto 10^{-18}$.
- 3) The corresponding experiment to observe such seldom events is suggested, which corresponds to detection of a macroscopic number of photons, $N \propto 10^8$.
- 4) This experiment can be the first one that would test some combination of general relativity and quantum mechanics on the Earth's orbit.
- 5) So far, only combinations of the Newton gravitation and Quantum Mechanics have been tested in the famous ILL and COW experiments.

- Widely advertised in the Internet:

  http://phys.org/news/2013-01-einstein-emc2-outer-space.html
  http://dailygalaxy.com/my_weblog/2013/01/-einsteins-emc2-may-breakdown-in-outer-space.html
  http://uanews.org/story/testing-einsteins-emc2-outer-space
  http://missiontomorrow.tv/testing-einsteins-emc2-in-outer-space-phys-org/
  http://www.sciencedaily.com/releases/2013/01/130108162227.htm

Metric theories of gravity

$$dS^2 = -\left[1 + 2\frac{\varphi}{c^2} + 2\beta\left(\frac{\phi}{c^2}\right)^2\right](cdt)^2 + \left[1 - 2\gamma\frac{\varphi}{c^2}\right](dx^2 + dy^2 + dx^2)$$

$$\hat{H} = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi$$

$$\hat{m}_e^g = m_e + \left(\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r}\right)/c^2 + \gamma\left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r}\right)/c^2$$

$$P_n(\gamma) = \gamma^2 * P_n(\gamma = 1)$$

$$N(n \to 1)(\gamma) = \gamma^2 * N(n \to 1)$$

The equivalence for superpositions of stationary quantum states survives only after averaging over time

$$\Psi_{1,2}(r,t) = \frac{1}{\sqrt{2}}\left[\Psi_1(r)\exp(-iE_1 t) + \Psi_2(r)\exp(-iE_2 t)\right], \qquad (25)$$

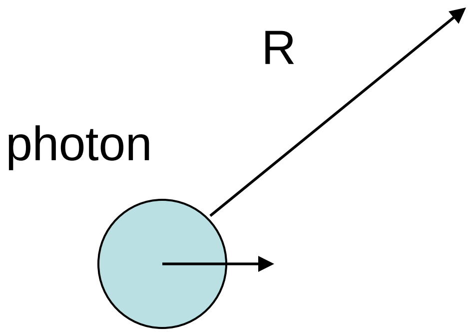
$$< \hat{m}_e^g \phi > = m_e \phi + \frac{E_1 + E_2}{2c^2}\phi + \frac{V_{1,2}}{c^2}\phi \cos\left[\frac{(E_1 - E_2)t}{\hbar}\right]. \qquad (26)$$

$$<< \hat{m}_e^g \phi >>_t = m_e \phi + \frac{(E_1 + E_2)}{2c^2}\phi. \qquad (27)$$

$$<< \hat{m}_e^g c^2 >>_t = m_e c^2 + (E_1 + E_2)/2$$

A.G. Lebed, arXiv: 1111.5365v1 [gr-qc] (2011); ibid. 1205.3134v1 [gr-qc] (2012); Marcel Grossmann Meeting-13 (arXiv: 1208.5657v1 [gr-qc] (2012)), to be published.

Active gravitational mass of a photon



$$\varphi(R) = -\frac{Gm_g}{R}$$ Tolman equation

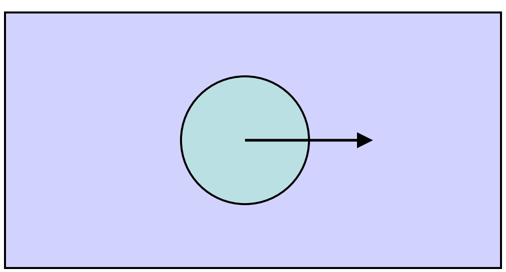
$$m_g = \int (T_0^0 - T_1^1 - T_2^2 - T_3^3)\sqrt{-g}\, d^3x$$

$$T_0^0 = -(T_1^1 + T_2^2 + T_3^3)$$

$$\downarrow$$

$$m_g = 2\int T_0^0 \sqrt{-g}\, d^3x = 2E/c^2$$

Container with mirrors



$$T_1^1 + T_2^2 + T_3^3$$

$$m_g = \int T_0^0 \sqrt{-g}\, d^3x = E/c^2$$

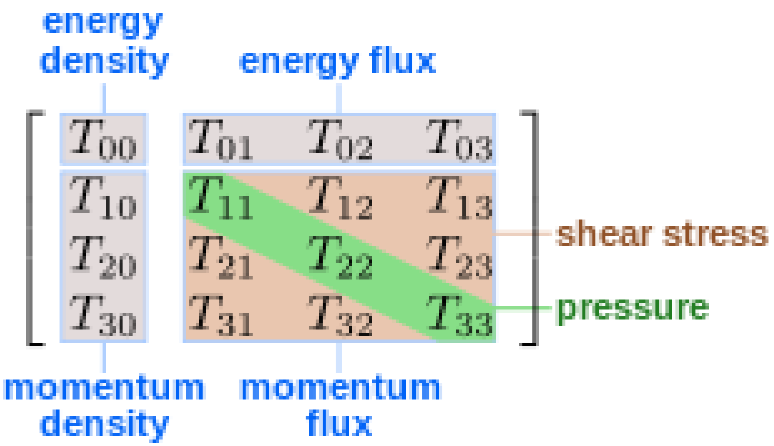
C.W. Misner and P. Putnam, Phys. Rev. 116, 1045 (1959)

Energy momentum tensor

For particle

$$T^{\alpha\beta}(\mathbf{x},t) = \frac{m\,v^\alpha(t) v^\beta(t)}{\sqrt{1-(v/c)^2}}\;\delta(\mathbf{x}-\mathbf{x}_\mathrm{p}(t)) = E\frac{v^\alpha(t)v^\beta(t)}{c^2}\;\delta(\mathbf{x}-\mathbf{x}_\mathrm{p}(t))$$

$$v^\alpha = \left(1, \frac{d\mathbf{x}_\mathrm{p}}{dt}(t)\right)\,,$$



For EM field

$$T^{\mu\nu}(x) = \frac{1}{\mu_0}\left(F^{\mu\alpha}g_{\alpha\beta}F^{\nu\beta} - \frac{1}{4}g^{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right)$$

E. Fischbach, B.S. Freeman, W.-K. Cheng,
Phys. Rev. D, v. 23, p. 2157 (1981): gravitational Stark effect

$$
\begin{aligned}
H(e-p) &= H^{(0)}(e-p) + \Delta H(e-p) \\
&= -\frac{Ze^2}{|\vec{r}|} + Mc^2 - M(c^2\Phi_E + \vec{g}\cdot\vec{R}) + \left[1-(2\gamma'+1)\Phi_E - (2\gamma'+1)\frac{\vec{g}\cdot\vec{R}}{c^2}\right]\left(\frac{\vec{P}^2}{2M} + \frac{\vec{k}^2}{2\mu_R}\right) \\
&\quad + (\gamma'+\tfrac{1}{2})\frac{i\hbar}{Mc^2}\vec{g}\cdot\vec{P} + \frac{\gamma'}{c^2}\left(\frac{1}{m_e} - \frac{1}{m_p}\right)(-\vec{g}\cdot\vec{r}\,\vec{k}^2 + i\hbar\,\vec{g}\cdot\vec{k}) + \frac{\gamma'}{c^2}\vec{g}\cdot\left(\frac{\vec{S}_e}{m_e} - \frac{\vec{S}_p}{m_p}\right)\times\vec{k} + \frac{(\gamma'+\frac{1}{2})}{Mc^2}\vec{g}\cdot\vec{S}\times\vec{P} \\
&\quad - \frac{\gamma'}{Mc^2}(\vec{g}\cdot\vec{r}\,\vec{P}\cdot\vec{k} + \vec{P}\cdot\vec{r}\,\vec{g}\cdot\vec{k} - i\hbar\,\vec{g}\cdot\vec{P}) + \frac{\gamma' Ze^2(m_p - m_e)}{2Mc^2}\frac{\vec{g}\cdot\vec{r}}{|\vec{r}|} + (\gamma'+1)\frac{Ze^2}{|\vec{r}|}\left(\Phi_E + \frac{\vec{g}\cdot\vec{R}}{c^2}\right).
\end{aligned}
$$

$$\hat{H} = \frac{\hat{p}^2}{2m_e} - \frac{e^2}{r} + \hat{m}_e^g \phi$$

$$\hat{m}_e^g = m_e + \left(\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r}\right)/c^2 + \gamma\left(2\frac{\hat{p}^2}{2m_e} - \frac{e^2}{r}\right)/c^2$$

From Schwarzschild solution

$$ds^2 = -\left(1+\frac{2\varphi}{c^2}\right)c^2dt^2 + \left(1+\frac{2\varphi}{c^2}\right)^{-1}d\hat{R}^2 - \hat{R}^2(d\theta^2 + \sin^2\theta d\Phi^2)$$

$$\varphi = -G\frac{M}{\hat{R}} \qquad \hat{R} = R\left(1+G\frac{M}{2Rc^2}\right)$$

$$ds^2 = -\left(1+\frac{\varphi}{2c^2}\right)^2\left(1-\frac{\varphi}{2c^2}\right)^{-2}c^2dt^2 + \left(1-\frac{\varphi}{2c^2}\right)^4(dx^2+dy^2+dz^2)$$

$$ds^2 = -\left(1+\frac{2\varphi}{c^2}\right)c^2dt^2 + \left(1-\frac{2\varphi}{c^2}\right)(dx^2+dy^2+dz^2)$$

$$\varphi = -G\frac{M}{R}$$

Classical Post-Newtonian Lagrangian

$$L = L_{KIN} + L_{EM} + L_G + L_{EG}$$

$$L_{KIN} = -\sum_i m_i \left(1 - \frac{1}{2} v_i^2 - \frac{1}{8} v_i^4\right) \qquad L_{EM} = -\frac{1}{2} \sum_{i,j} \frac{e_i e_j}{r_{ij}}$$

$$L_G = \frac{G}{2} \sum_{i,j} \frac{m_i m_j}{r_{i,j}} \left[1 - \frac{1}{2}\left(\vec{v}_i * \vec{v}_j + \vec{v}_i * \vec{r}_{ij} \vec{v}_j * \vec{r}_{ij}\right)\right] + \frac{3}{4} \sum_{ij} \frac{m_i m_j}{r_{ij}} v_{ij}^2$$

$$L_{EG} = \frac{G}{2} \sum_{ijk} \frac{e_i e_j}{r_{ij}} \left(\frac{m_k}{r_{ik}} + \frac{m_k}{r_{jk}}\right)$$

$$v << \alpha c$$

K. Nordtvedt, Class. Quantum Grav., 11, A119 (1994)