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Passive scalar transport in peripheral regions of random flows

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Landau Days

We designate the passive scalar field as θ.
It can represent both, temperature variations or concentration of pollutants. The passive scalar evolution in an external flow is described by the equation

$$\partial_t\theta + v\nabla\theta = \kappa\nabla^2\theta\,,$$

where v is the flow velocity and κ is the diffusion (thermodiffusion) coefficient. The coefficient is assumed to be small, $\kappa \ll \nu$.

In bulk the mixing time is estimated as λ^{-1} where λ is the Lyapunov exponent, irrespective to the κ value. However, the mixing time near walls is sensitive to the κ value, it can be estimated as $\sqrt{\nu/\kappa}\,\lambda^{-1}$. Besides, the velocity correlation time is estimated as λ^{-1} even near the wall. Thus, the velocity can be treated as short correlated in time.

If the velocity is short correlated then closed equations can be derived for the passive scalar correlation functions

$$F_n(t, \boldsymbol{r}_1, \ldots, \boldsymbol{r}_n) = \langle \theta(t, \boldsymbol{r}_1) \ldots \theta(t, \boldsymbol{r}_n) \rangle \,,$$

obtained by averaging over times larger than the velocity correlation time. Let us stress that the situation is strongly anisotropic.

One derives the following equations

$$\partial_t F_n = \kappa \sum_{m=1}^{n} \nabla_m^2 F_n$$
$$+ \sum_{m,k=1}^{n} \sum_{\alpha\beta} \partial_{m\alpha} \left[D_{\alpha\beta}(\boldsymbol{r}_m, \boldsymbol{r}_k) \partial_{k\beta} F_n \right] ,$$

where the object D is expressed via the pair velocity correlation function as

$$D_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_0^\infty dt \, \langle v_\alpha(t, \boldsymbol{r}_1) v_\beta(0, \boldsymbol{r}_2) \rangle \, .$$

A z-dependence of the eddy diffusion tensor components can be found directly from the proportionality laws $v_x, v_y \propto z$ and $v_z \propto z^2$. Say,

$$D_{zz}(x, y, z_1; x, y, z_2) = \mu z_1^2 z_2^2 \,,$$

where μ is a constant characterizing strength of the velocity fluctuations in the peripheral region.

The equation for the first moment of θ is

$$\partial_t \langle\theta\rangle = \partial_z \left[\mu z^4 \partial_z \langle\theta\rangle\right] + \kappa \partial_z^2 \langle\theta\rangle \,,$$

Comparing two terms in RHS, one finds a characteristic diffusion length

$$r_{bl} = (\kappa/\mu)^{1/4}.$$

The quantity determines the thickness of the diffusion boundary layer.

We are interested mainly in the passive scalar transport through the region $z \gg r_{bl}$, where the passive scalar is carrying from the diffusive boundary layer to bulk. There we arrive at the proportionality law

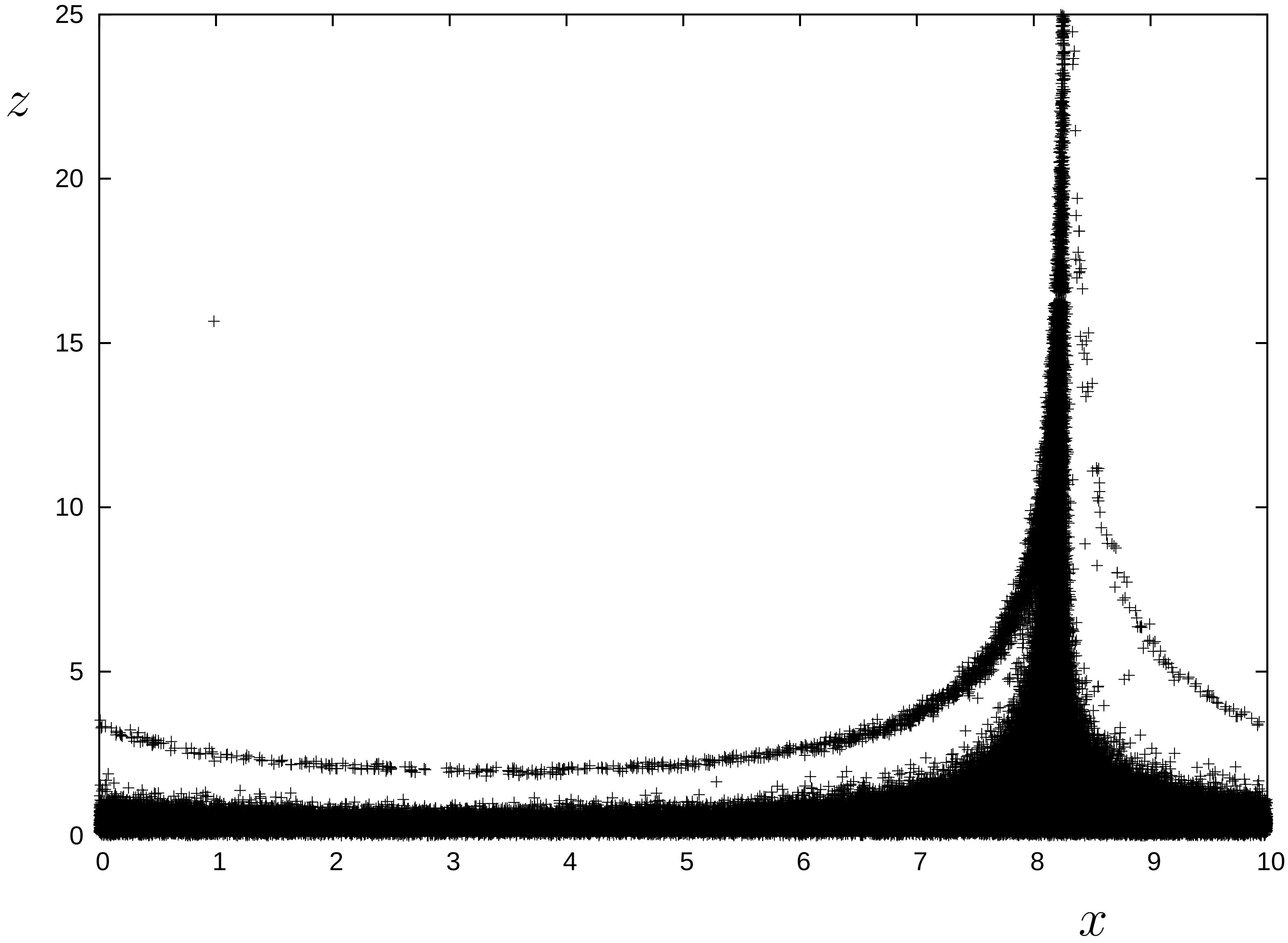
$$\langle \theta \rangle \propto z^{-3},$$

that gives the decaying rate of the average θ as z grows.

We introduce scaling exponents for the high passive scalar moments as well

$$\langle \theta^n \rangle \propto z^{-\eta_n},$$

at $z \gg r_{bl}$. Would the molecular diffusion be irrelevant there then $\eta_n = 3$. Really, the diffusion is relevant and values of the exponents η_n are subject of a special investigation.



25
20
15
10
5
0
z
0
1
2
3
4
5
6
7
8
9
10
x

One can define the passive scalar correlation length l (along the wall), that can be found by balance of the molecular and the eddy diffusion along the wall:

$$l \sim \sqrt{\kappa/\mu}\; z^{-1}.$$

The quantity is of order of r_{bl} at $z \sim r_{bl}$ and diminishes as z grows.

To exclude the effect of the molecular diffusion, we introduce an integral of the passive scalar field

$$\Theta(t,z) = A^{-1}\int dx\ dy\ \theta(t,x,y,z)\,,$$

where A is the area of the surface and z is its separation from the wall. Obviously $\langle\Theta\rangle \propto z^{-3}$. What about high-order moments?

Assuming that the passive scalar correlation length is smaller than the velocity one, we can derive

$$\partial_t \Phi_n(t, z_1, \ldots, z_n) = \mu \sum_{m,k=1}^{n} \frac{\partial}{\partial z_m} \left(z_m^2 z_k^2 \frac{\partial}{\partial z_k} \Phi_n \right)$$

$$+ 2\mu \sum_{m \neq k} \frac{\partial}{\partial z_m} \left(z_m^2 z_k \Phi_n \right) ,$$

$$\Phi_n(t, z_1, \ldots, z_n) = \langle \Theta(t, z_1) \ldots \Theta(t, z_n) \rangle .$$

The equation leads to the following closed equation for the moments of the integral passive scalar

$$\partial_t \langle \Theta^n \rangle = \mu \left[z^4 \partial_z^2 + 4nz^3 \partial_z + 4n(n-1)z^2 \right] \langle \Theta^n \rangle \,.$$

The equation leads to the scaling

$$\langle \Theta^n \rangle \propto z^{-\zeta_n}, \qquad \zeta_n = 2n - 1/2 + \sqrt{2n + 1/4}\,.$$

A natural conjecture that enables one to relate the moments of θ and those of $\boldsymbol{\Theta}$ is in using the correlation length l as a recalculation factor:

$$\langle \boldsymbol{\Theta}^n \rangle \sim \frac{l^{(d-1)(n-1)}}{A^{n-1}} \langle \theta^n \rangle,$$

$$\eta_n = \zeta_n - (n-1)(d-1).$$

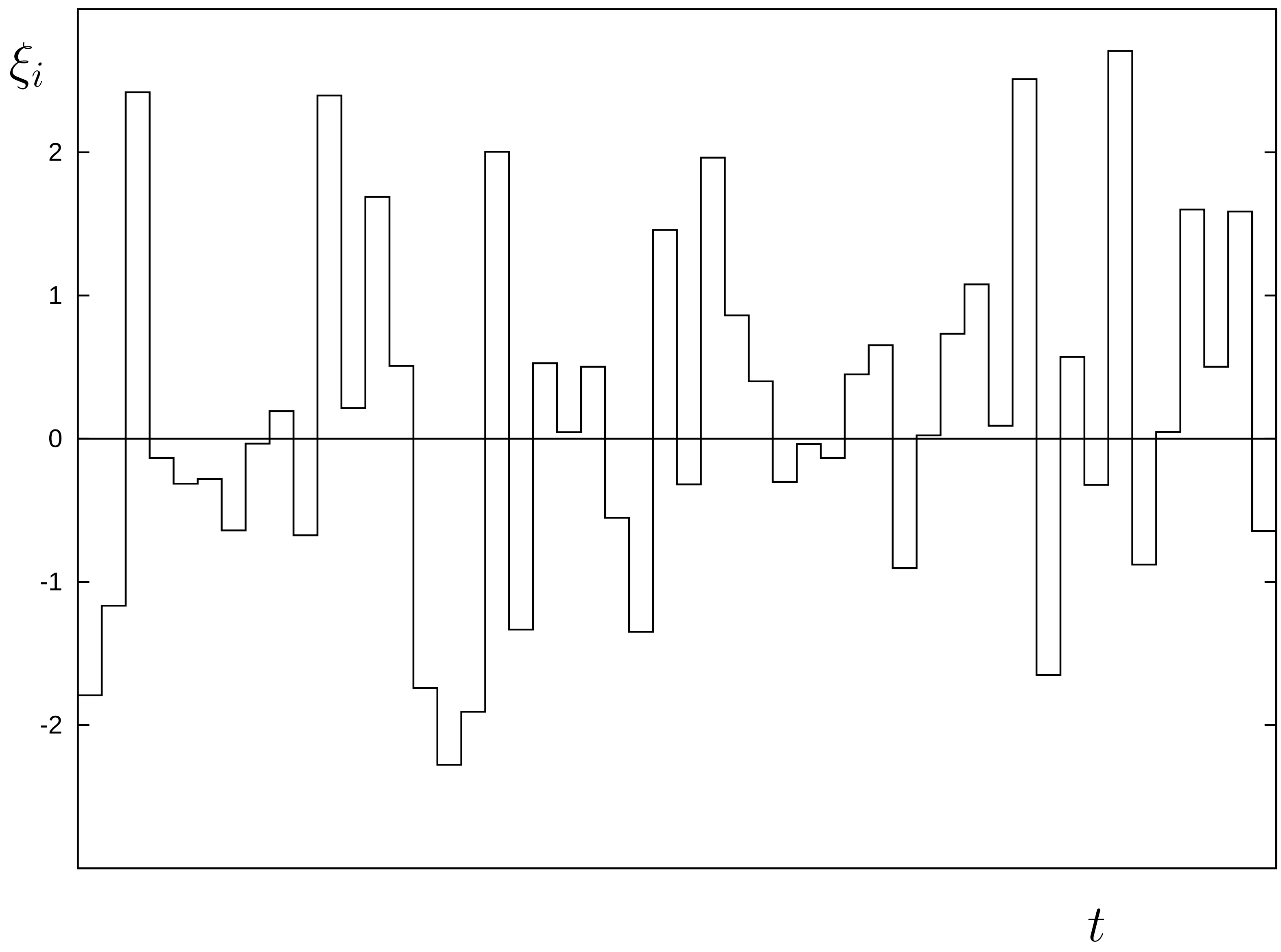
Here d is dimensionality of space.

We conducted Lagrangian simulations where dynamics of a large number of particles subjected to flow advection and Langevin forces is examined. The set of the particles is used instead of the passive scalar field θ, that can be treated as density of the particles. A big advantage of the approach is its applicability to a number of space dimensions d.

In our scheme a particle trajectory $\varrho(t)$ obeys the equation

$$\partial_t \varrho = v(t, \varrho) + \chi(t),$$

where the first term represents the particle advection and the second term represents the Langevin force. The variables χ are independent for different particles whereas the velocity is the same.

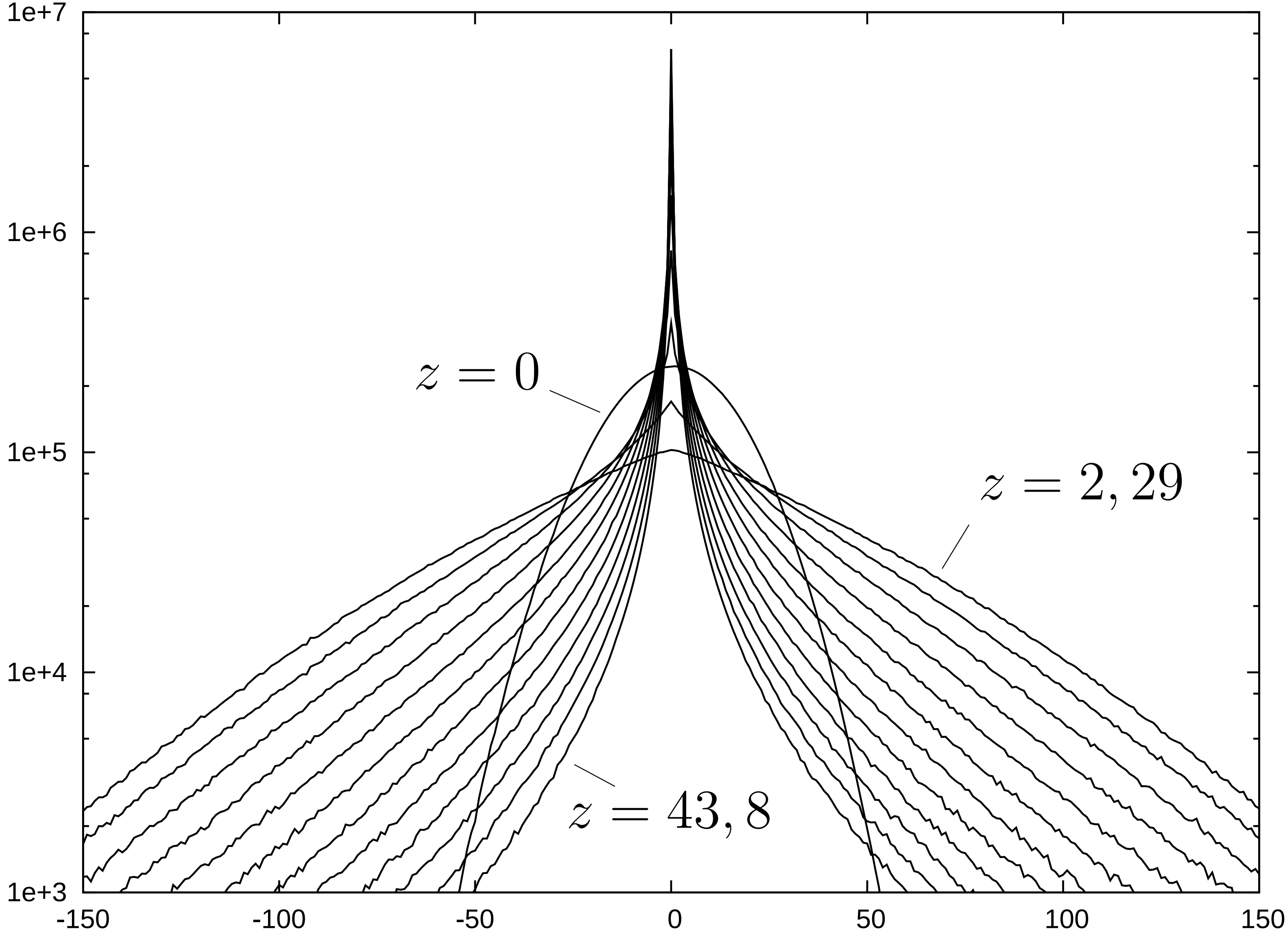


ξ_i
2
1
0
-1
-2
t

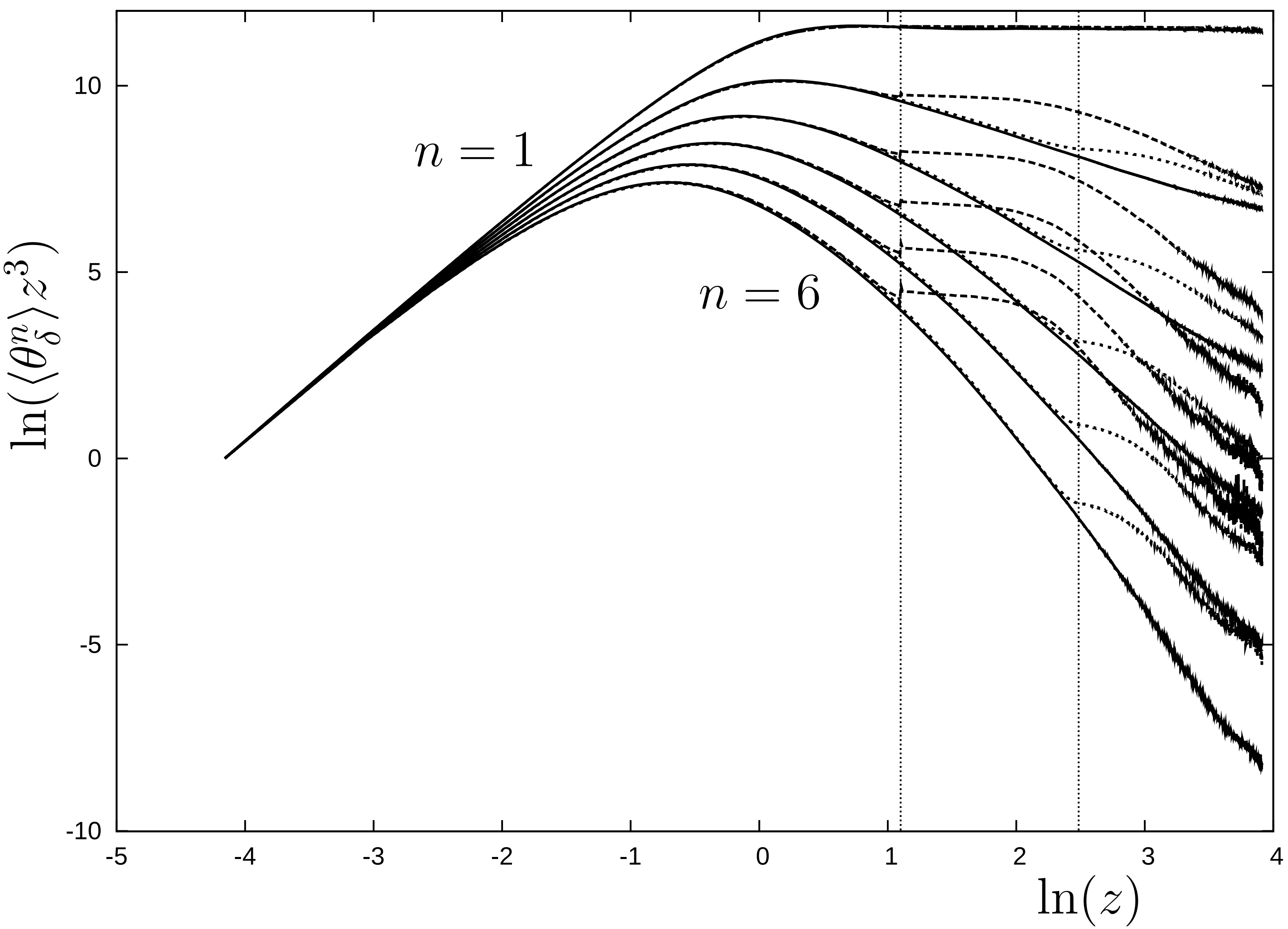
To establish principal qualitative features of the process, we perform mainly $2d$ simulations. The setup is periodic in x and the velocity in majority of runs was

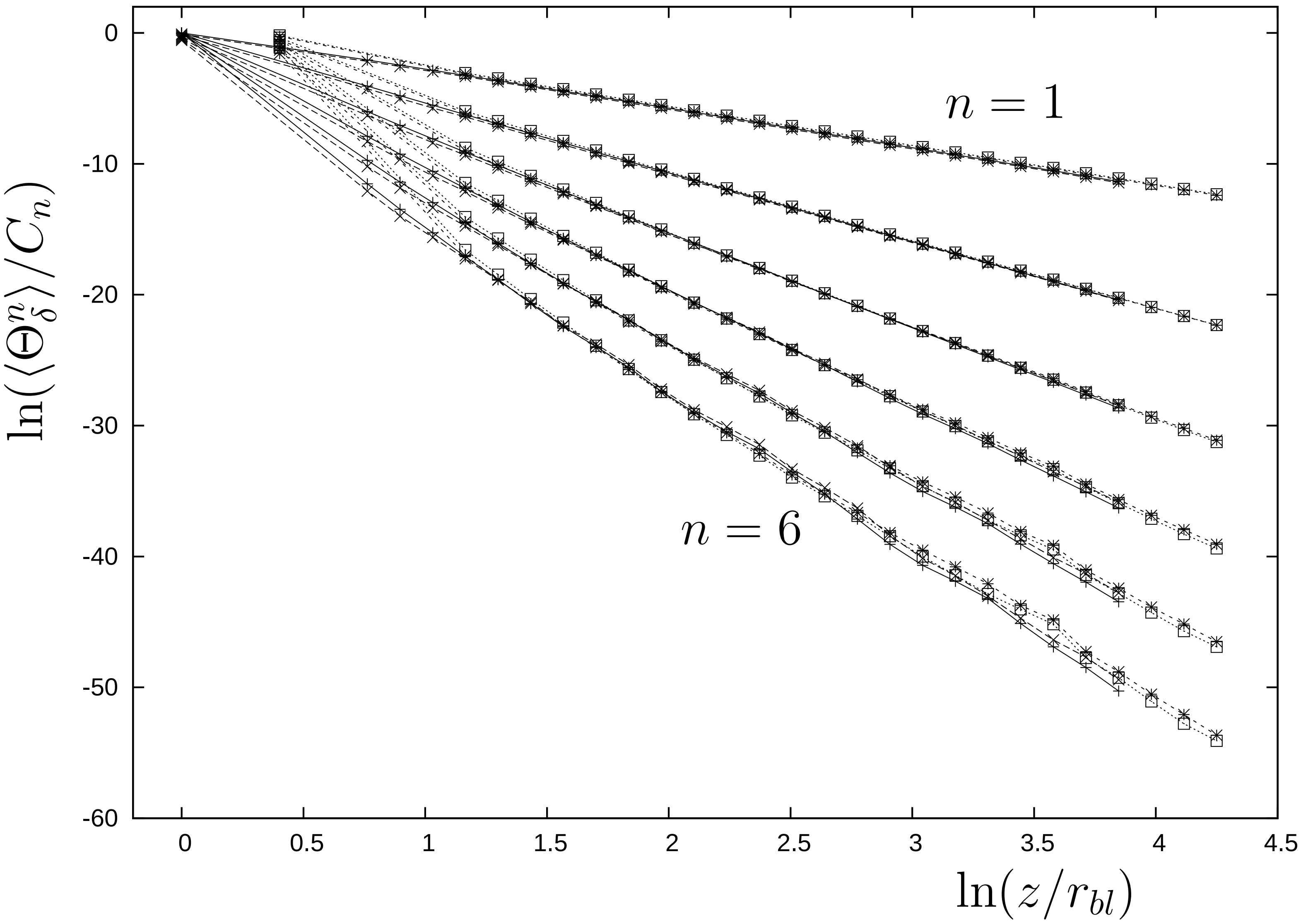
$$v_x = z\left(\xi_1 \cos\frac{2\pi x}{L} + \xi_2 \sin\frac{2\pi x}{L}\right)\frac{L}{\pi},$$
$$v_z = z^2\left(\xi_1 \sin\frac{2\pi x}{L} - \xi_2 \cos\frac{2\pi x}{L}\right),$$

where ξ_1 and ξ_2 are independent random functions of time.

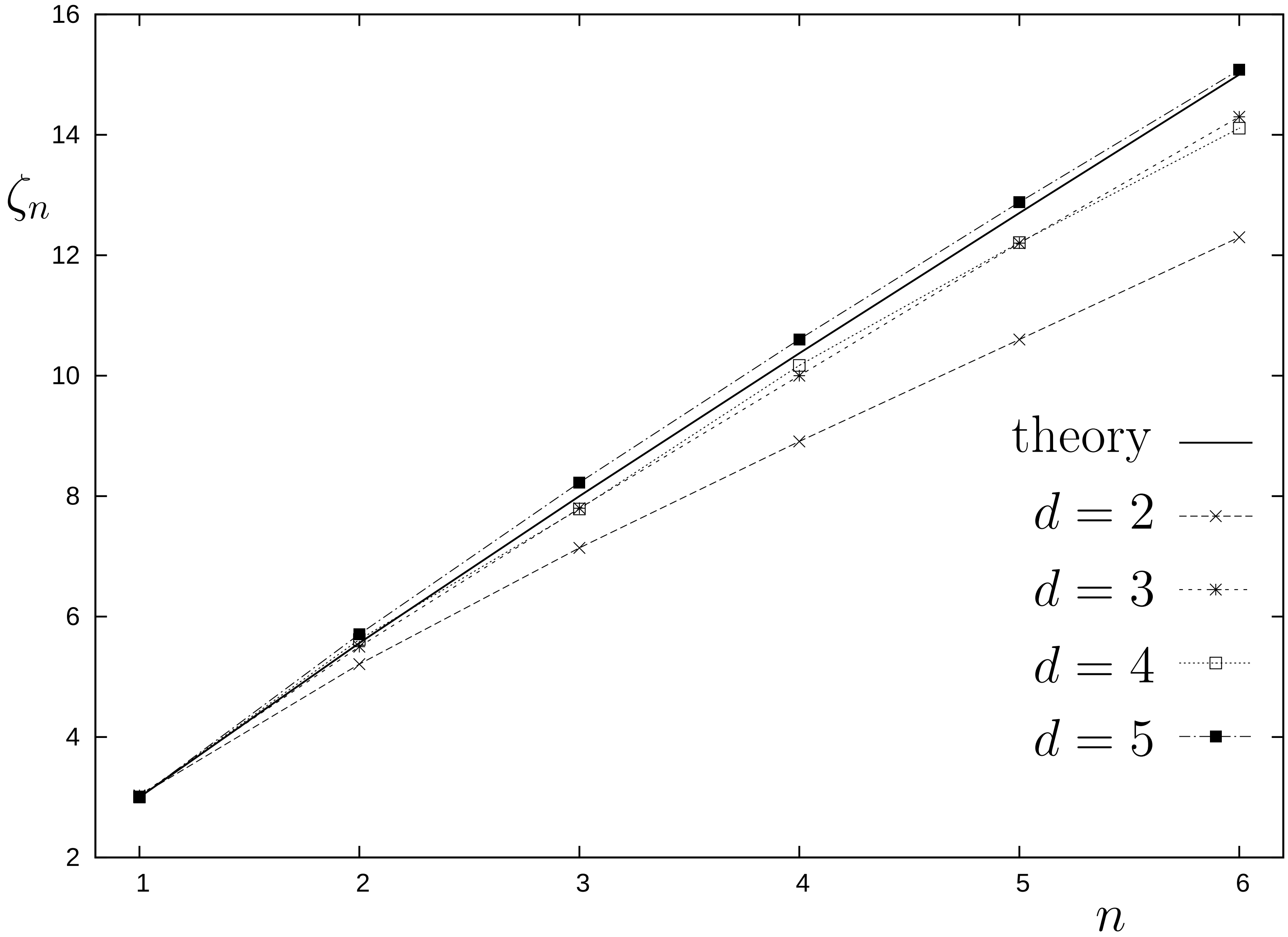


1e+7
1e+6
1e+5
1e+4
1e+3
-150
-100
-50
0
50
100
150
$z = 0$
$z = 2,29$
$z = 43,8$


$n = 1$
$n = 6$
10
5
0
-5
-10
-5
-4
-3
-2
-1
0
1
2
3
4
$\ln(\langle \theta_\delta^n \rangle z^3)$
$\ln(z)$



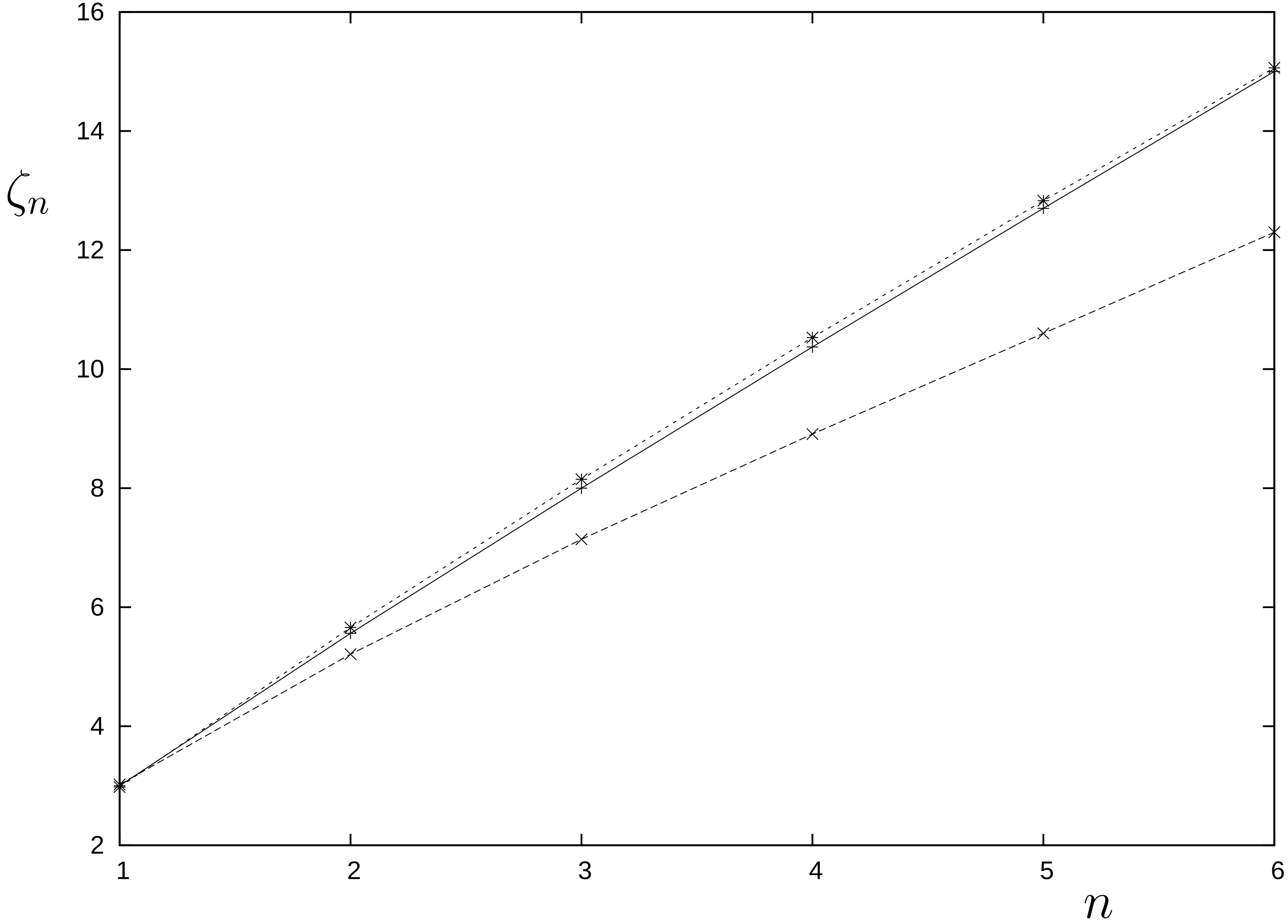
$\ln(\langle \Theta_\delta^n \rangle / C_n)$
$\ln(z/r_{bl})$
$n = 1$
$n = 6$
0
-10
-20
-30
-40
-50
-60
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5



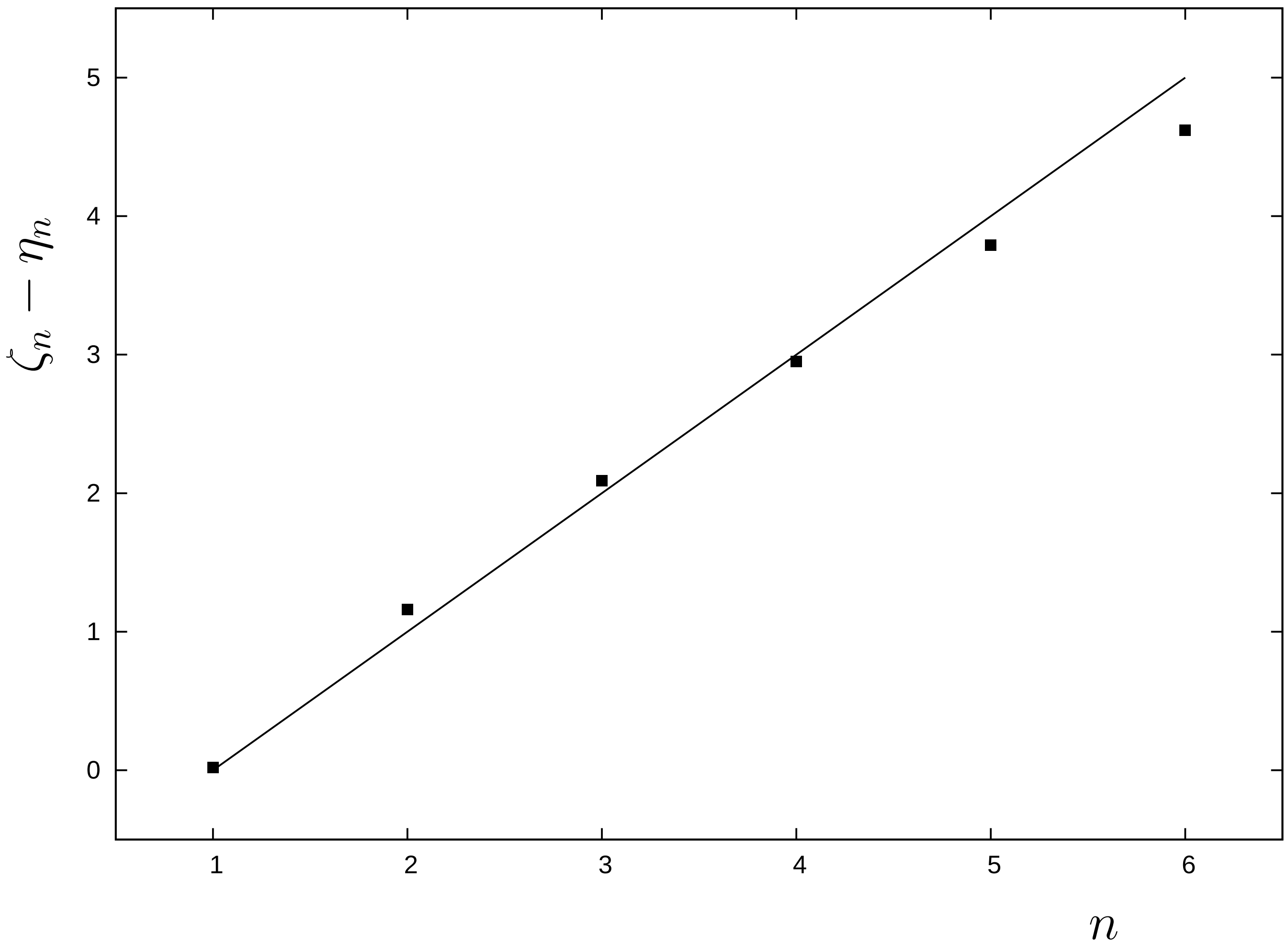
16
14
12
10
8
6
4
2
ζ_n
1
2
3
4
5
6
n
theory
$d=2$
$d=3$
$d=4$
$d=5$

Numerics reveal deviations of the scaling exponents from the analytical predictions that are related to an existence of long correlations along the wall that can be produced by the multi-fold structures. That leads to increasing moments in comparison with the short correlated case. That is why the scaling exponents are smaller than the theoretical values.

The deviations naturally diminish as d grows. However, the effect is a consequence of the artificial fact that the velocity correlation length coincides with the size system. Let us make the velocity correlation length smaller than L by using a mixture of the ninth and the eleventh harmonics. Then we arrive at a good agreement with numerics.



16
14
12
10
8
6
4
2
ζ_n
1
2
3
4
5
6
n



$\zeta_n - \eta_n$
0
1
2
3
4
5
1
2
3
4
5
6
n

Inertial particles are governed by the equation

$$\tau \frac{d\boldsymbol{u}}{dt} + \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\xi}.$$

Here, $\boldsymbol{u}$ is the particle velocity, $\boldsymbol{v}$ is the flow velocity, $\boldsymbol{\xi}$ is Langevin force and τ is the particle relaxation time associated with its inertial properties.

If the diffusion is neglected then the equation for the one-particle probability density is

$$\partial_t \rho = -u\partial_z \rho + \partial_u (u\rho) + z^4 \partial_u^2 \rho,$$

where we put $\mu = \tau = 1$. We will be looking for stationary distribution of the probability density that corresponds to zero probability flux to z-infinity:

$$\int_{-\infty}^{+\infty} du \ u\rho = 0.$$

For $u \ll z$ the equation is reduced to

$$\partial_u(u\rho) + z^4 \partial_u^2 \rho = 0,$$

that have the solution

$$\rho \propto z^{-6} \exp\left(-\frac{u^2}{2z^4}\right).$$

It is realized at $z \ll 1$ since $u \sim z^2$.

For $u \gg z$ the equation is reduced to

$$u \partial_z \rho = z^4 \partial_u^2 \rho.$$

After the self-similar substitution $\rho = z^{-5a} h(\xi)$ where $\xi = u/z^{5/3}$ we obtain

$$h'' + (5/3)\xi^2 h' + 5a\xi h = 0.$$

It can be solved in terms of the confluent hypergeometric function.

Both asymptotics are zero if

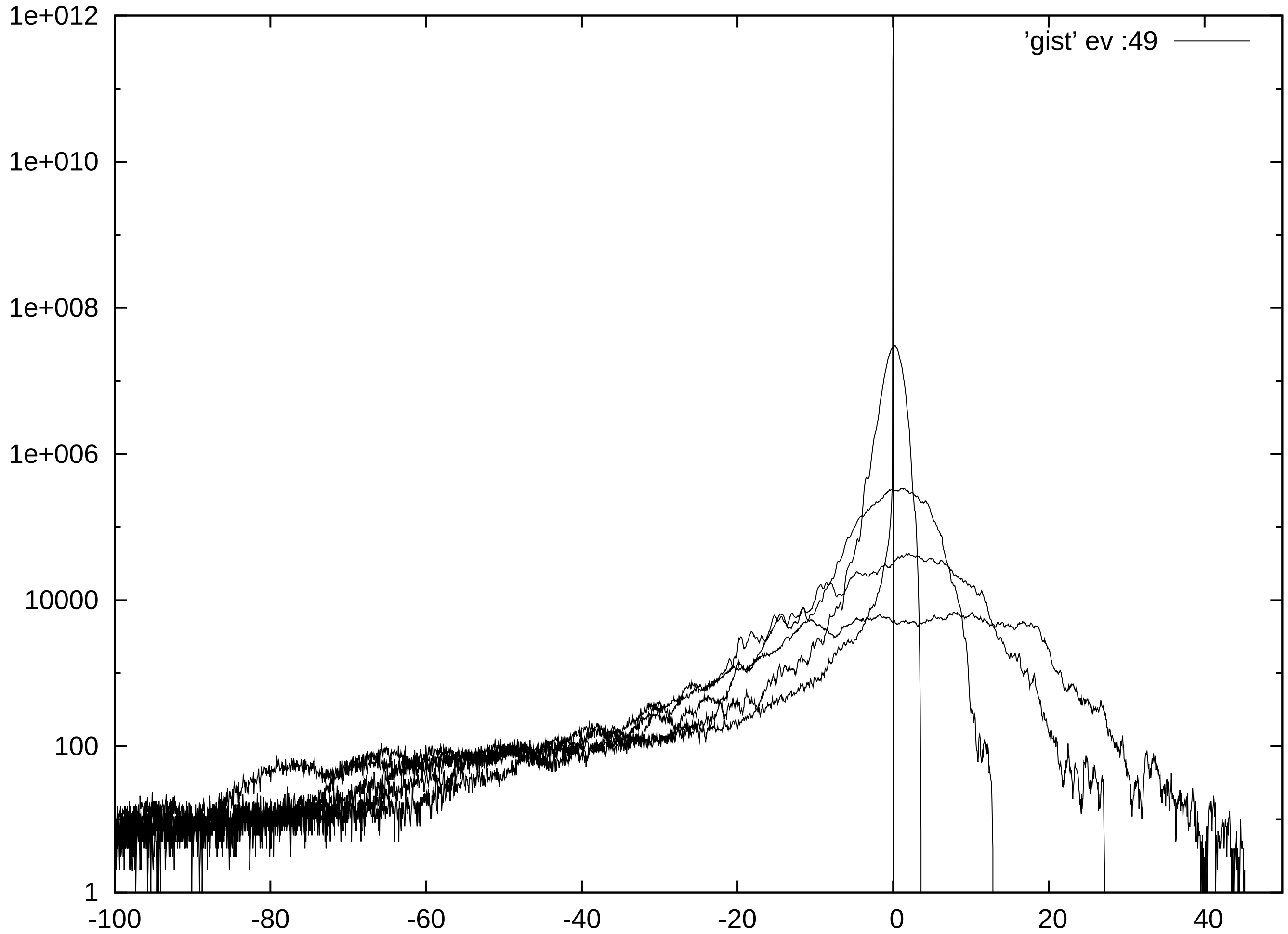
$$a = -\frac{1}{6} + n \to 5/6.$$

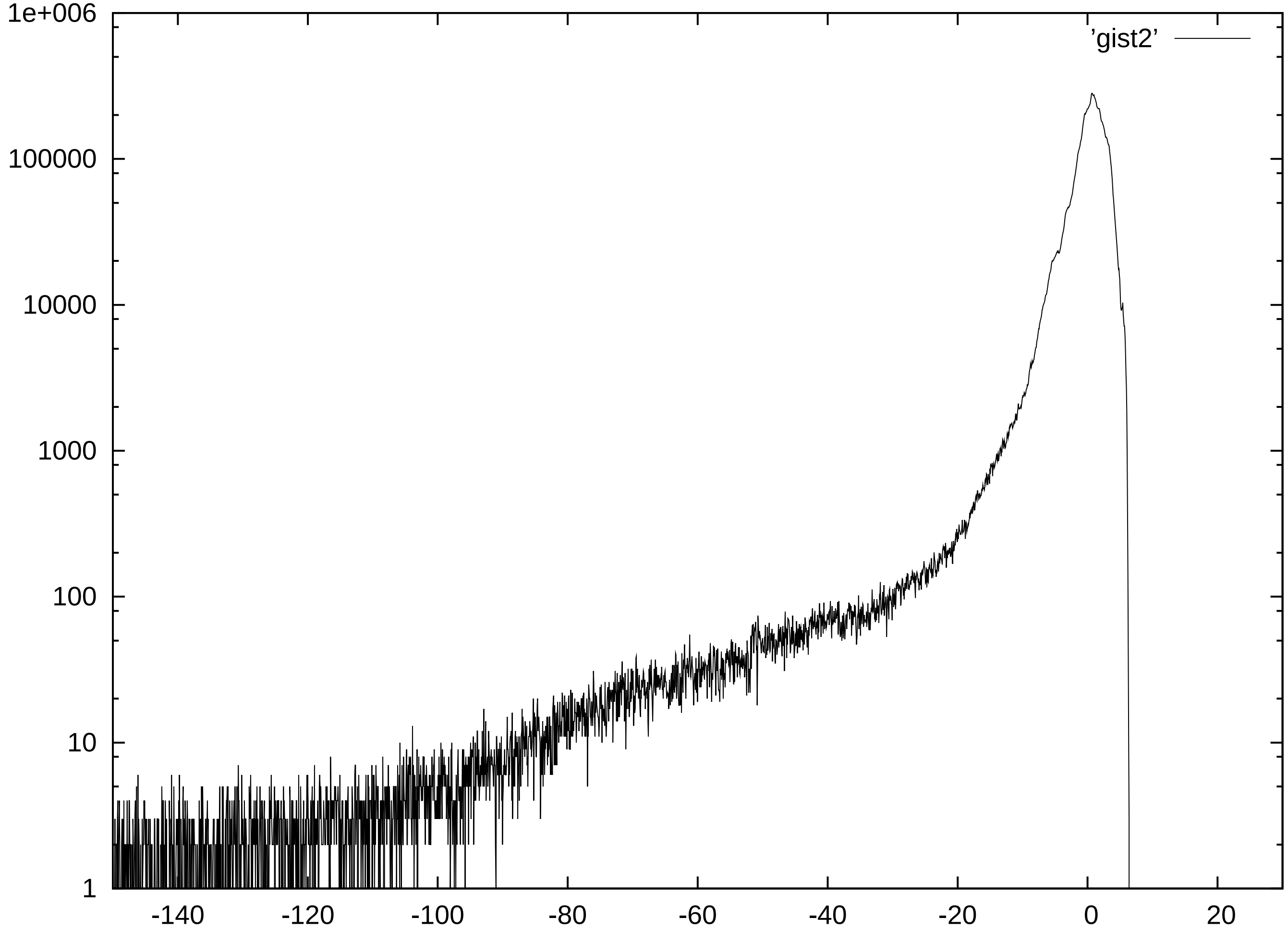
Then at large negative u: $\rho \propto |u|^{-5/2}$.

At large positive u

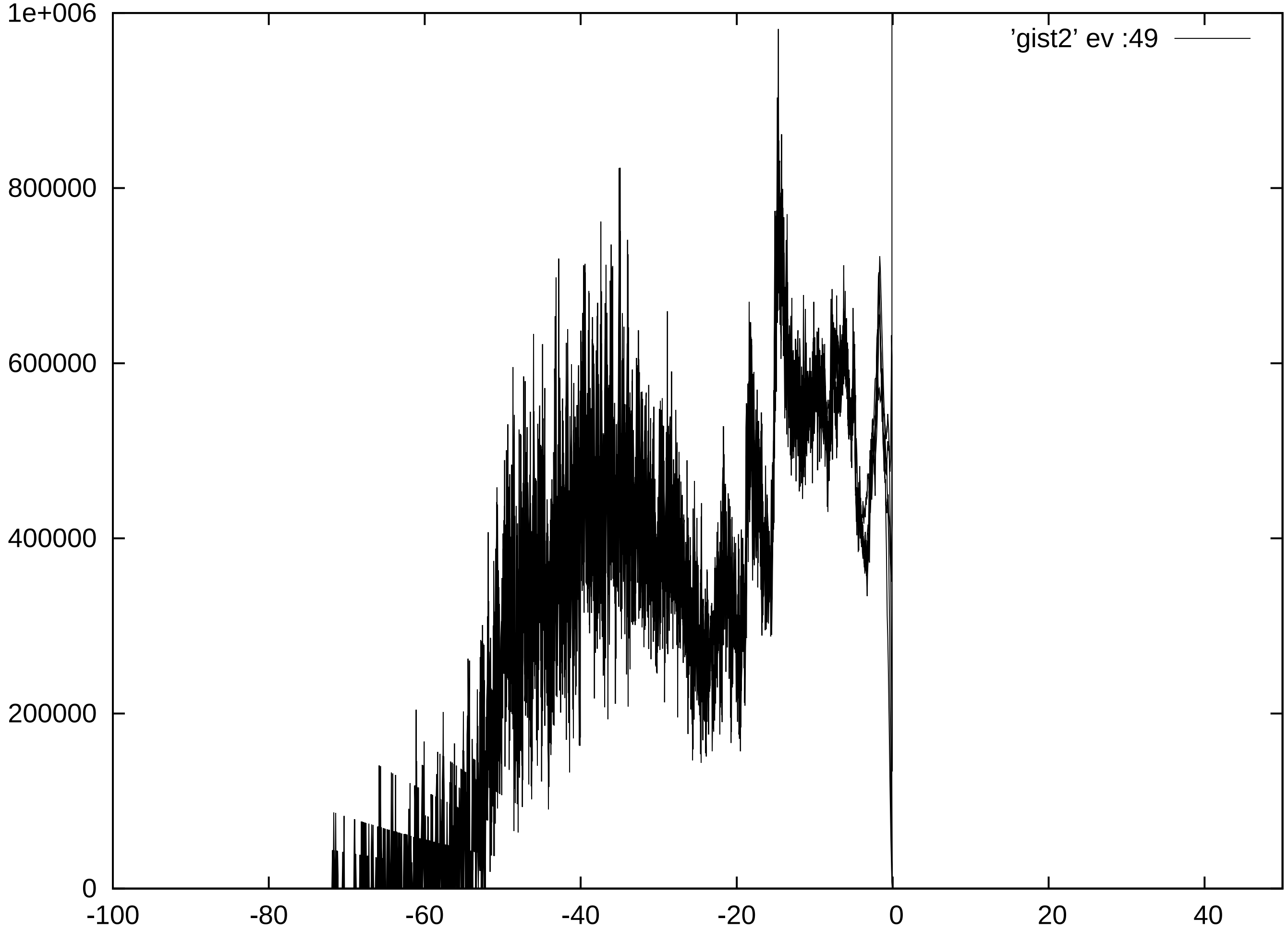
$$\rho \propto \exp\left(-\frac{5}{9}\frac{u^3}{z^5}\right).$$

Highly non-symmetric distribution.


1e+012
1e+010
1e+008
1e+006
10000
100
1
-100
-80
-60
-40
-20
0
20
40
'gist' ev :49



'gist2'
1e+006
100000
10000
1000
100
10
1
-140
-120
-100
-80
-60
-40
-20
0
20



1e+006
800000
600000
400000
200000
0
-100
-80
-60
-40
-20
0
20
40
'gist2' ev :49