Russian Academy of Sciences

Landau Institute for Theoretical Physics

Multifractality at Anderson transitions with Coulomb interaction

Igor Burmistrov

in collaboration with

Igor Gornyi (Karlsruhe Inst. of Tech., Germany)

Alexander Mirlin (Karlsruhe Inst. of Tech., Germany)

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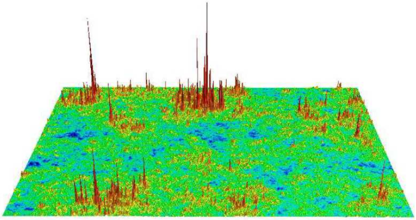
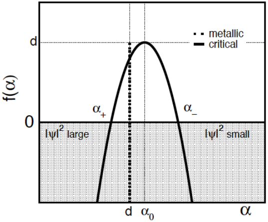
[Wegner (1980,1987); Kravtsov, Lerner (1985); Pruisken(1985); Castellani, Peliti (1986)]

$$\int\limits_{|\boldsymbol{r}|\leqslant L} d^d\boldsymbol{r} \left\langle \left|\psi_E(\boldsymbol{r})\right|^{2q} \right\rangle_{\mathrm{dis}} \sim L^{-\tau_q}, \quad \tau_q = \begin{cases} d(q-1), & \text{metal,} \\ d(q-1)+\Delta_q, & \text{criticality,} \\ 0, & \text{insulator} \end{cases}$$

- multifractal exponent $\Delta_q \leqslant 0$ ($\Delta_1 = 0$ due to w.f. normalization)
- Legendre transform of τ_q: $f(\alpha) = q\alpha - \tau_q, \quad \alpha = d\tau_q/dq$
- $L^{f(\alpha)}$ measures a set of points where $\left|\psi_E\right|^2 \sim L^{-\alpha}$



[adapted from Evers, Mildenberger, Mirlin]



[adapted from Evers, Mirlin (2008)]

- local density of states

$$\rho(E, \boldsymbol{r}) = \sum_{\alpha} \left|\psi_\alpha(\boldsymbol{r})\right|^2 \delta(E - \epsilon_\alpha)$$

where $\psi_\alpha(\boldsymbol{r})$ and ϵ_α w. f. and energy for a given disorder

- multifractality in LDOS moments

$$\left\langle [\rho(E, \boldsymbol{r})]^q \right\rangle_{\text{dis}} \sim L^{-\Delta_q}$$

- spatial correlations of LDOS

$$\langle \rho(E, \boldsymbol{r}) \rho(E, \boldsymbol{r} + \boldsymbol{R}) \rangle_{\text{dis}} \sim (R/L)^{\Delta_2} \qquad R \ll L$$

examples for Anderson transitions in $d = 2$:

$\Delta_2 = -0.34$ (class AII)
$\Delta_2 = -0.52$ (class A, integer qHe)
$\Delta_2 = -1/4$ (class C, spin qHe)

[see for a review, Evers&Mirlin (2008)]

[Altshuler, Aronov, Lee(1980), Finkelstein(1983), Castellani, DiCastro, Lee, Ma(1984)]
[Nazarov (1989), Levitov, Shytov (1997), Kamenev, Andreev (1999)]

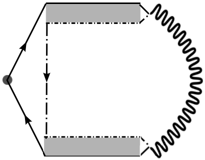
- zero-bias anomaly in $d = 2$

$$\left\langle \rho(E, \boldsymbol{r}) \right\rangle_{\text{dis}} \sim \exp\left(-\frac{1}{4\pi g} \ln\bigl(|E|\tau\bigr) \ln \frac{|E|}{D^2\kappa^4\tau} \right)$$

where g - conductance in units e^2/h, D - diffusion coefficient,
$\kappa = e^2\rho_0/\epsilon$ - inverse static screening length

- zero-bias anomaly in $d = 2 + \epsilon$ at Anderson transitions

$$\left\langle \rho(E, \boldsymbol{r}) \right\rangle_{\text{dis}} \sim |E|^{\beta}, \qquad \beta = O(1)$$



in the absence of interaction average LDOS is non-critical ($\beta = 0$) for Wigner-Dyson classes
[see for a review, Finkelstein (1990), Kirkpatrick&Belitz (1994)]

[Gruzberg, Read, Ludwig (1999), Beamond, Cardy, Chalker (2002)]
[Mirlin, Evers, Mildenberger (2003)]

- average LDOS ($L = \infty$)

$$\left\langle \rho(E, \boldsymbol{r}) \right\rangle \sim |E - E_c|^{1/7}, \qquad E_c = 0$$

- multifractality in the moments of LDOS

$$\left\langle [\rho(E, \boldsymbol{r})]^q \right\rangle_{\text{dis}} \sim \left\langle [\rho(E, \boldsymbol{r})] \right\rangle_{\text{dis}}^q \mathcal{L}^{-\Delta_q}, \quad \mathcal{L} = \min\{L, \xi(E)\}$$

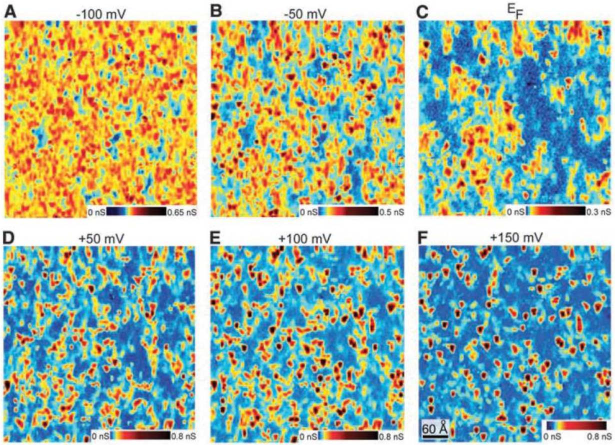
where localization/correlation length $\xi(E) \sim |E - E_c|^{-4/7}$,
$\Delta_2 = -1/4$, $\Delta_3 = -3/4$

- spatial correlations of LDOS

$$\left\langle \rho(E, \boldsymbol{r}) \rho(E, \boldsymbol{r} + \boldsymbol{R}) \right\rangle_{\text{dis}} \sim \left\langle [\rho(E, \boldsymbol{r})] \right\rangle_{\text{dis}}^2 (R/\mathcal{L})^{\Delta_2} \qquad R \ll \mathcal{L}$$

[Richardella et al. (2010)]

- differential conductance over an area of 500 Å× 500 Å in $Ga_{1-x}Mn_xAs$ with $x = 1.5\%$

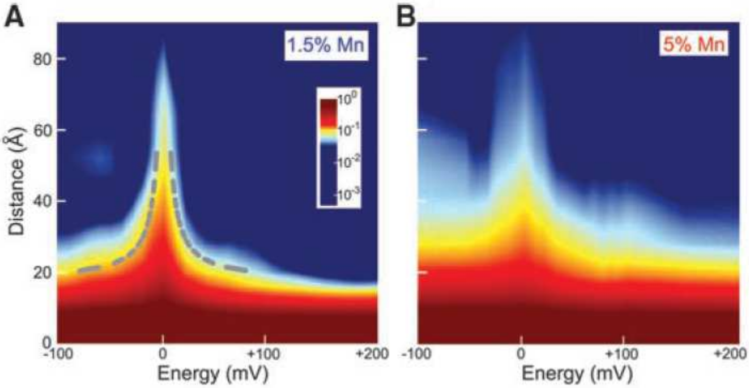


mean-free path $l = 10$ Å

[Richardella et al. (2010)]

- LDOS correlation function

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle - \langle \rho(E,\boldsymbol{r})\rangle^2}{\langle \rho^2(E,\boldsymbol{r})\rangle^2 - \langle \rho(E,\boldsymbol{r})\rangle^2}$$

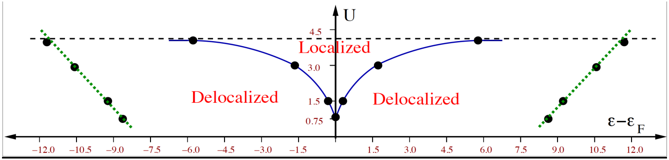


[Amini, Kravtsov, Müller (2013)]

- LDOS correlation function made from Hartree-Fock w.f.

$$\frac{\langle \rho_{HF}(E,\boldsymbol{r})\rho_{HF}(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho_{HF}^2(E,\boldsymbol{r})\rangle^2}$$

- extracted phase diagram interaction vs energy



How Coulomb interaction affects mesoscopic fluctuations of the local density of states?

- free electrons

$$H_0 = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\left[-\frac{\nabla^2}{2m}\right]\psi_\sigma(\boldsymbol{r})$$

- scattering off white-noise random potential

$$H_{\rm dis} = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\,V(\boldsymbol{r})\psi_\sigma(\boldsymbol{r}), \qquad \langle V(\boldsymbol{r})\,V(0)\rangle = \frac{1}{2\pi\rho_0\tau}\delta(\boldsymbol{r})$$

- scattering off magnetic impurities

$$H_{\rm mag} = \int d^d\boldsymbol{r}\,\overline{\psi}_\sigma(\boldsymbol{r})\,\boldsymbol{U}(\boldsymbol{r})\boldsymbol{\sigma}_{\sigma\sigma'}\psi_{\sigma'}(\boldsymbol{r}), \qquad \langle U_a(\boldsymbol{r})\,U_b(0)\rangle = \frac{\delta_{ab}}{6\pi\rho_0\tau_s}\delta(\boldsymbol{r})$$

- Coulomb interaction:

$$H_{\rm int} = \frac{1}{2}\int d^2\boldsymbol{r_1}d^2\boldsymbol{r_2}\,\frac{e^2}{\epsilon|\boldsymbol{r_1}-\boldsymbol{r_2}|}\,\overline{\psi}_\sigma(\boldsymbol{r_1})\psi_\sigma(\boldsymbol{r_1})\overline{\psi}_{\sigma'}(\boldsymbol{r_2})\psi_{\sigma'}(\boldsymbol{r_2})$$

$H_0 + H_{\rm dis} + H_{\rm mag}$ do not preserve time-reversal and spin-rotational symmetry $\Rightarrow$ class A

- disorder-averaged moments of the LDOS

$$\left\langle \left[\rho(E,\boldsymbol{r})\right]^q \right\rangle_{\text{dis}} = \left\langle \left[-\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{G}^R(E,\boldsymbol{r},\boldsymbol{r})\right]^q \right\rangle_{\text{dis}}$$

where the single-particle Green function

$$\mathcal{G}^R(\boldsymbol{r}t;\boldsymbol{r'}t') = -i\theta(t-t')\left\langle \left\{\overline{\psi}(\boldsymbol{r}t),\psi(\boldsymbol{r'}t')\right\} \right\rangle$$

- assumptions

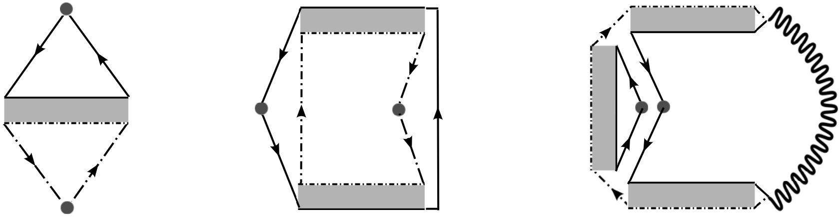
$$\mu \gg \frac{1}{\tau} \gg \frac{1}{\tau_s} \gg T, |E|$$

where

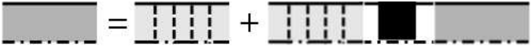
μ – the chemical potential
τ – the mean-free time for scattering off potential impurities
τ_s – the mean-free time for scattering off magnetic impurities
T – the temperature
E – the energy measured from the chemical potential

we consider $T=0$ in what follows

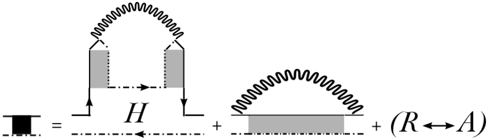
- one- and two-loop contributions to the 2d moment of LDOS



- diffuson with self-energy due to interaction



- diffuson self-energy due to interaction



actually, 2 loop calculations in the non-linear σ model approach

[Abrahams, Anderson, Licciardello, Ramakrishnan (1979), Wegner (1980)]
[Efetov, Larkin, Khmelnitsky (1980)]

- RG equation for conductivity $\sigma = (e^2/h)L^{2-d}/(\pi t)$:

$$-\frac{dt}{d\ln L} = \beta(t)$$

- fixed point t_* and localization/correlation length exponent ν:

$$\beta(t_*) = 0, \qquad \xi \propto \left|t - t_*\right|^{-\nu}, \qquad \nu = -1/\beta'(t_*)$$

- dynamical exponent z:

$$L_\omega \sim \omega^{-1/z}$$

in general, there is another dynamical exponent z_T, $L_\phi \sim T^{-1/z_T}$

[Hikami (1983), Bernreuther&Wegner (1986)]
[Castellani, DiCastro, Lee, Ma (1984), Finkelstein(1984)]
[Baranov, Pruisken, Škorić (1999), Baranov, Burmistrov, Pruisken (2002)]

- no interaction

$$\beta_{\rm n}(t) = \epsilon t - \frac{1}{2}t^3 - \frac{3}{8}t^5 + O(t^6)$$

critical conductance

$$t_*^{(\rm n)} = \sqrt{2\epsilon}\left(1 - \frac{3\epsilon}{4}\right) + O(\epsilon^{5/2})$$

critical exponents

$$\nu_{\rm n} = \frac{\epsilon}{2} - \frac{3}{4} + O(\epsilon)$$
$$z_{\rm n} = d = 2 + \epsilon$$
$$\beta = 0$$

- Coulomb interaction

$$\beta(t) = \epsilon t - 2t^2 - 4At^3 + O(t^4)$$

$$t_* = \frac{\epsilon}{2}(1 - A\epsilon) + O(\epsilon^3)$$

$$\nu = \frac{1}{\epsilon} - A + O(\epsilon)$$
$$z = 2 - \frac{\epsilon}{2} + \left(2A - \frac{\pi^2}{6} - 3\right)\frac{\epsilon^2}{4} + O(\epsilon^3)$$
$$\beta = \frac{1}{2} + O(\epsilon)$$

where

$$A = \frac{139}{96} + \frac{(\pi^2 - 18)^2}{192} + \frac{19}{32}\zeta(3) + \left(1 + \frac{\pi^2}{48}\right)\ln^2 2 - \left(44 - \frac{\pi^2}{2} + 7\zeta(3)\right)\frac{\ln 2}{16} + \mathcal{G} - \frac{1}{48}\ln^4 2 - \frac{1}{2}\,\mathrm{li}_4\left(\frac{1}{2}\right) \approx 1.64$$

$$\left\langle [\rho(E,\boldsymbol{r})]^q \right\rangle \sim \langle \rho(E) \rangle^q \mathcal{L}^{-\Delta_q} \sim \mathcal{L}^{-\beta z q - \Delta_q} \qquad \mathcal{L} = \min\{L_E, L, \xi\}$$

where multifractal exponents and anomalous dimensions are

$$\Delta_q = \zeta_q(t_*) = \frac{q(1-q)\epsilon}{4}\left[1 + \left(1 - A - \frac{\pi^2}{12}\right)\epsilon\right] + O(\epsilon^3)$$

$$\zeta_q(t) = \frac{q(1-q)t}{2}\left[1 + \left(2 - \frac{\pi^2}{6}\right)t\right] + O(t^3)$$

cf. non-interacting electrons

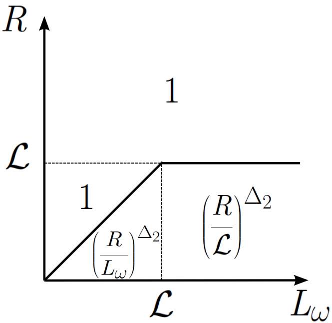
$$\Delta_q^{(\mathrm{n})} = \zeta_q^{(0)}(t_*) = q(1-q)\left(\frac{\epsilon}{2}\right)^{1/2} - \frac{3\zeta(3)}{32} q^2(q-1)^2\epsilon^2 + O(\epsilon^{5/2})$$

$$\zeta_q^{(\mathrm{n})}(t) = \frac{q(1-q)t}{2}\left(1 + \frac{3t^2}{8} + \frac{3\zeta(3)}{16} q(q-1)t^3\right) + O(t^5),$$

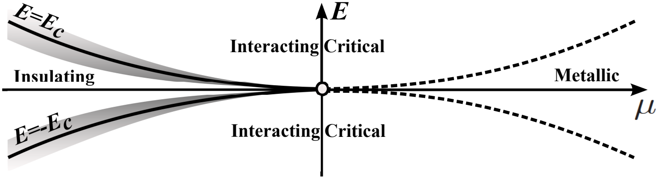
[Höf&Wegner (1986), Wegner (1987)]

- spatial and energy correlations of LDOS ($\mathcal{L} = \min\{L_E, L, \xi\}$)

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E+\omega,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle\langle\rho(E+\omega)\rangle} \sim \begin{cases} (R/L_\omega)^{\Delta_2}, & R \ll L_\omega \ll \mathcal{L} \\ 1, & L_\omega \ll R, \mathcal{L} \\ (R/\mathcal{L})^{\Delta_2}, & R \ll \mathcal{L} \ll L_\omega \\ 1, & \mathcal{L} \ll R, L_\omega \end{cases}$$

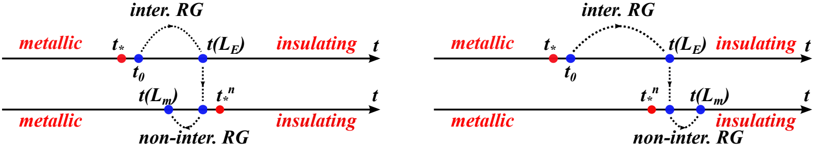


- phase diagram near interacting critical point $\mu = \mu_c$ $(t = t_*)$



- mobility edge for single particle excitations $E = \pm E_c$,

$$E_c \sim (t - t_*)^{\nu z} \sim (\mu_c - \mu)^{\nu z}$$



- divergent localization and dephasing lengths

$$\xi(E) \sim \big||E| - E_c\big|^{-\nu_n}, \qquad L_\phi(E) \sim \begin{cases} \big(|E| - E_c\big)^{-z_n}, & |E| > E_c \\ \infty, & |E| < E_c \end{cases}$$

preliminary result: $z_n = d^2/[2(d-1)]$

Spatial correlations of LDOS / insulating phase $t > t_*$

- interacting criticality $|E| \gg E_c$

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim \begin{cases} (R/L_E)^{\Delta_2}, & R \ll L_E \sim |E|^{-1/z} \\ 1, & L_E \ll R \ll \xi \sim E_c^{-1/z} \end{cases}$$

- non-interacting criticality $\big||E| - E_c\big| \ll E_c$
  - above mobility edge $|E| > E_c$

$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim \begin{cases} (R/\xi)^{\Delta_2} (\xi/L_\phi(E))^{\Delta_2^{(\mathrm{n})}}, & R \ll \xi \\ (R/L_\phi(E))^{\Delta_2^{(\mathrm{n})}}, & \xi \ll R \ll L_\phi(E) \end{cases}$$

  - below mobility edge $|E| < E_c$

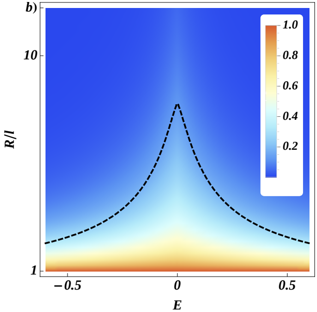
$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim \begin{cases} (R/\xi)^{\Delta_2} (\xi/\xi(E))^{\Delta_2^{(\mathrm{n})}} (L/\xi(E))^d, & R \ll \xi \\ (R/\xi(E))^{\Delta_2^{(\mathrm{n})}} (L/\xi(E))^d, & \xi \ll R \ll \xi(E) \end{cases}$$

- deep below the mobility edge $|E| \ll E_c$

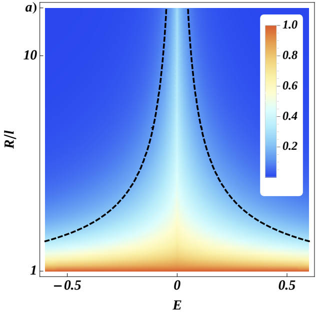
$$\frac{\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle}{\langle \rho(E)\rangle^2} \sim (R/\xi)^{\Delta_2} (L/\xi)^d, \quad R \ll \xi$$

where $\xi(E) \sim \big||E| - E_c\big|^{-\nu_\mathrm{n}}$, $L_\phi(E) \sim \big(|E| - E_c\big)^{-1/z_\mathrm{n}}$. It is assumed that $z_\mathrm{n}\nu_\mathrm{n} > 1$

$$\frac{\langle\langle \rho(E,\boldsymbol{r})\rho(E,\boldsymbol{r}+\boldsymbol{R})\rangle\rangle}{\langle\langle \rho^2(E,\boldsymbol{r})\rangle\rangle} \sim \begin{cases} (R/\mathcal{L})^{-\Delta_2}, & R \ll \mathcal{L} = \min\{|E|^{-1/z}, (\mu-\mu_c)^{-1/\nu}\} \\ 0, & \mathcal{L} \ll R \end{cases}$$



metallic phase $t < t_*$ $(\mu > \mu_c)$



critical phase $t = t_*$ $(\mu = \mu_c)$

qualitatively in agreement with the experiment

Conclusions

- multifractality of LDOS does exist in the interacting disordered electron systems
- the multifractal exponents and corresponding anomalous dimensions are different from the non-interacting case