Water waves and analytical structure of Stokes waves

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3D Euler's equations of incompressible fluid motion in gravitational field *g*:

$$\mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} + \frac{1}{\rho}\nabla p + \mathbf{g} = 0$$

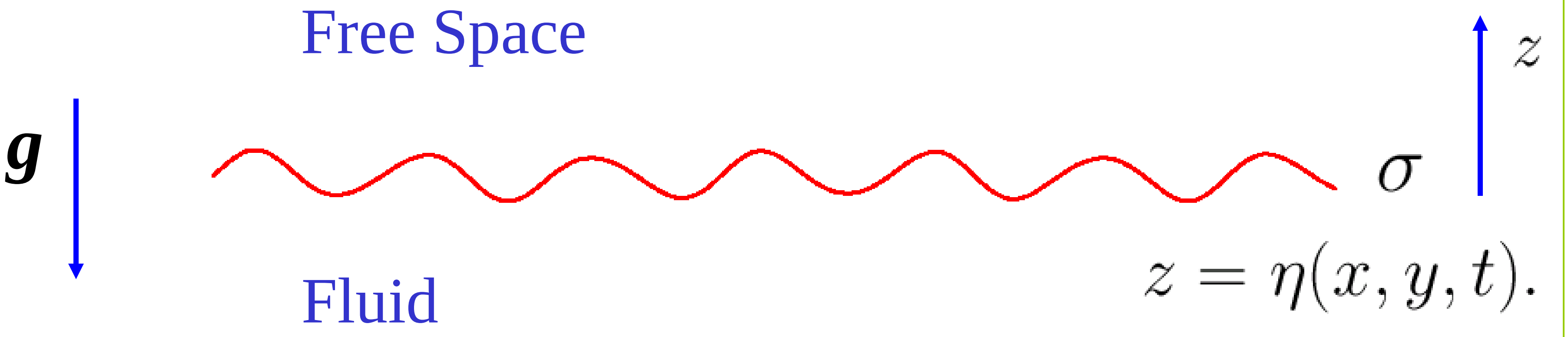
$$\nabla\cdot\mathbf{v} = 0$$

Reduction: potential flow

$$\mathbf{v} = \nabla\phi \quad \Rightarrow \quad \nabla\cdot\mathbf{v} = \Delta\phi = 0$$ - Laplace Eq.

$$\nabla\left[\phi_t + \frac{(\nabla\phi)^2}{2} + \frac{1}{\rho}p + gz\right] = 0$$ - Bernouilli Eq.

Free surface hydrodynamics



Laplace Eq.: $\Delta\Phi = 0$

$\boldsymbol{g}$ - acceleration of gravity

σ - surface tension coefficient

$z = \eta(x, y, t)$ - shape of free surface

$\Phi_z|_{z=-h} = 0$ - boundary condition at the bottom

Boundary conditions at free surface:

Kinematic condition:

$$\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + (\vec{V}\nabla)\eta = V_z$$

Dynamic boundary condition:

$$p\Big|_{z=\eta} = \sigma \nabla \cdot \frac{\nabla \eta}{\sqrt{1 + (\nabla \eta)^2}}$$

$p\Big|_{z=\eta}$ - pressure at free surface $z = \eta(x, y, t).$

Bernouilli Eq.:

$$\Phi_t + \frac{1}{2}\left(\nabla \Phi\right)^2 + p + gz = 0$$

Kinematic and dynamic boundary conditions together with Laplace Eqs. $\Delta\Phi = 0$ form a closed set of equations.

Equivalent Hamiltonian formulation (Zakharov, 1968):

$$\frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta},$$
$$\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi},$$

where $\Psi \equiv \Phi\big|_{z=\eta}$ - velocity potential at free surface

The Hamiltonian =kinetic energy+ potential energy, $H = T + U$

$$T = \frac{1}{2}\int d\mathbf{r} \int_{-h}^{\eta} \left(\nabla\Phi\right)^2 dz,$$

$$U = \frac{1}{2} g \int \eta^2 \, d\mathbf{r} + \sigma \int \left[\sqrt{1 + \left(\nabla\eta\right)^2} - 1\right] d\mathbf{r}$$

$\frac{1}{2} g \int \eta^2 \, d\mathbf{r}$ - potential energy in the gravitational field

$\int \left[\sqrt{1 + \left(\nabla\eta\right)^2} - 1\right] d\mathbf{r}$ - surface area

The Hamiltonian be rewritten as a surface integral:

$$H = \frac{1}{2}\int \Big[V_n \Psi \sqrt{1+(\nabla\eta)^2} + g\eta^2 + 2\sigma(\sqrt{1+(\nabla\eta)^2} - 1)\Big]\, d^2\mathbf{r}$$

$$\mathbf{r} = (x, y),\ \nabla = (\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}).$$

Normal velocity component: $V_n = \mathbf{n} \cdot \nabla\Phi$

Unit normal vector: $\mathbf{n} = (-\nabla\eta, 1)\dfrac{1}{\sqrt{1+(\nabla\eta)^2}}$

The Hamiltonian perturbation theory:

The Hamiltonian H has to be expressed in terms of canonical variables Ψ and η which requires to solve the Laplace equation $\triangle\Phi = 0$ with Dirichlet boundary condition $\Psi \equiv \Phi\big|_{z=\eta}$
Or, in other words, it is necessary to determine Dirichlet-Neumann operator

$$\hat{G}\Psi = \left[1 + (\nabla\eta)^2\right]^{1/2} \mathbf{n} \cdot \nabla\Phi|_{z=\eta}$$

which determines normal derivative of potential Φ from boundary data $\Psi \equiv \Phi\big|_{z=\eta}$

Perturbation technique:

Series expansion of V_n in powers of Ψ and η.

Small parameter of perturbation theory: $|\nabla\eta|$ - a typical slope of surface elevation.

Free Surface Hydrodynamics for 2D Flow at Infinite depth without surface tension:

Free surface: $y = \eta(x)$

Complex variable: $z = x + iy$

Conformal map to half-plane: $z \longrightarrow w = u + iv$

$$y = \eta(x) \longrightarrow v = 0$$

Fluid dynamics in conformal variables[1]:

$$y_t = (y_u \hat{H} - x_u)\frac{\hat{H}\Psi_u}{|z_u|^2} \qquad z = x + iy$$

$$\Psi_t = \frac{\hat{H}(\Psi_u \hat{H}\Psi_u)}{|z_u|^2} + \Psi_u \hat{H}\left(\frac{\hat{H}\Psi_u}{|z_u|^2}\right) - gy + \alpha\frac{1}{x_u}\frac{\partial}{\partial u}\frac{y_u}{|z_u|}$$

$$x = u - \hat{H}y$$

Hilbert transform: $$\hat{H}f(x) = \frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{f(x')}{x'-x}dx'$$

$$\hat{k} = -\frac{\partial}{\partial x}\hat{H}$$

Hilbert transform in Fourier domain: $-i\,\mathrm{sign}(k)$

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[1]A.I. Dyachenko et. al., Phys. Lett. A **221**, 73 (1996).

$$x = u + \tilde{x}(u, t)$$

$$\tilde{z} = \tilde{x} + iy$$

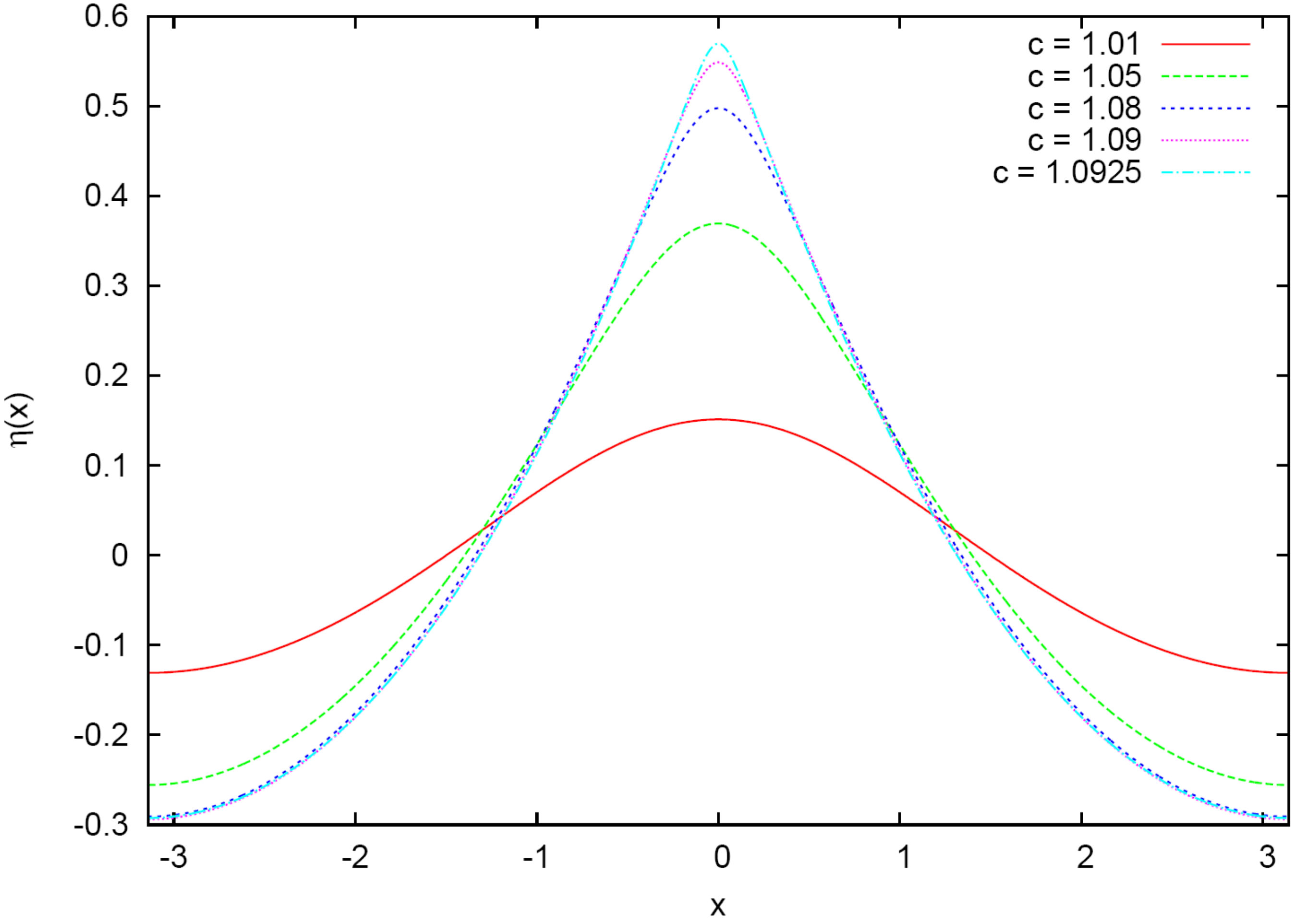
Progressive wave (Stokes wave)

$$\tilde{z} = \tilde{z}(u - ct)$$
$$\psi = \psi(u - ct)$$

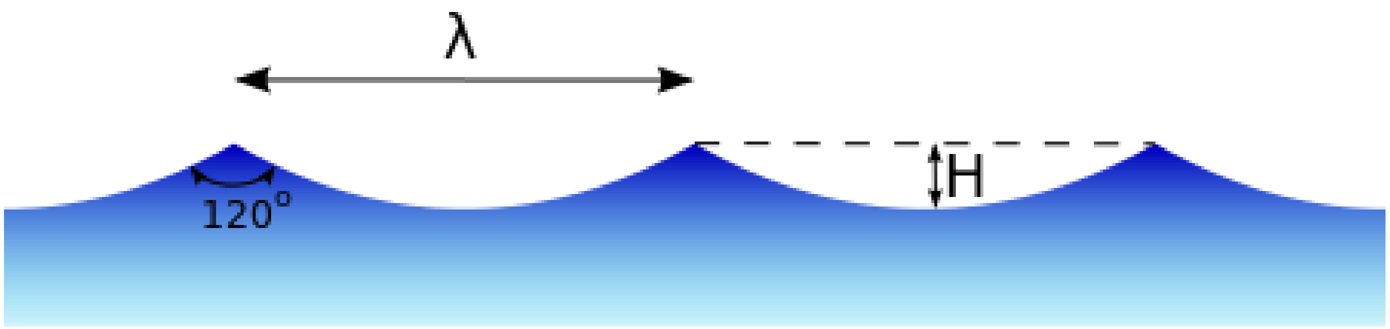
$$\Rightarrow \qquad \boxed{z_u = 1 + \frac{2g}{c^2}\hat{P}[yz_u]}$$

$\hat{P} = \frac{1+i\hat{H}}{2}$ - Projector to a function analytic in lower half plane

Stokes wave for different velocities *c* with *g=1*



Stokes wave of maximum height



$$H / \lambda \approx 0.1412$$

Low amplitude limit of Stokes wave

$$\eta(x,t) = a\left\{\cos\theta + \tfrac{1}{2}(ka)\,\cos 2\theta + \tfrac{3}{8}(ka)^2\,\cos 3\theta\right\} + O\left((ka)^4\right),$$

$$\phi(x,y,t) = a\frac{\omega}{k}\,\mathrm{e}^{ky}\,\sin\theta + O\left((ka)^4\right),$$

$$c = \frac{\omega}{k} = \left(1 + \tfrac{1}{2}(ka)^2\right)\,\sqrt{\frac{g}{k}} + O\left((ka)^4\right)$$

$$\theta(x,t) = kx - \omega t,$$

Water waves are not integrable (fourth order matrix element is zero while 5th order is not zero on resonance surfaces.

Instead we suggest to look for the dynamics of complex singularities

Second conformal transform

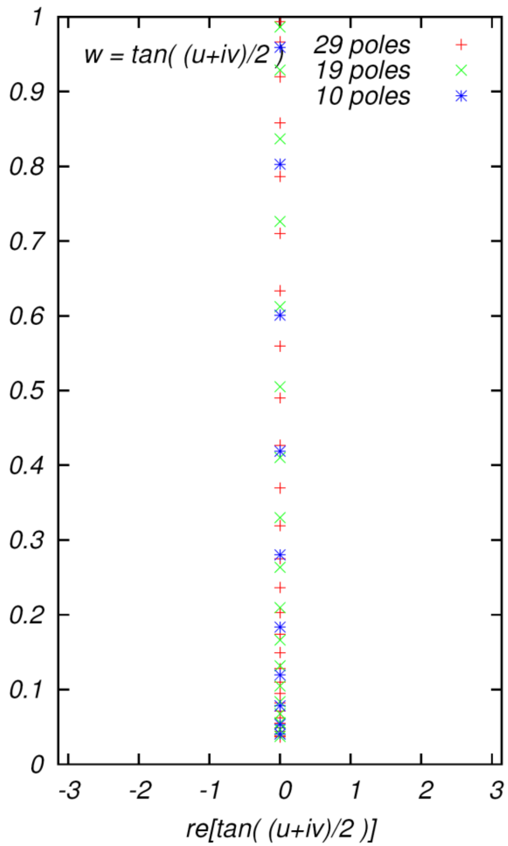
$$\zeta = \kappa + i\chi = \tan(\tfrac{w}{2})$$

Maps $u \in [-\pi, \pi]$ to the real line $\kappa \in (-\infty, \infty), \chi = 0$

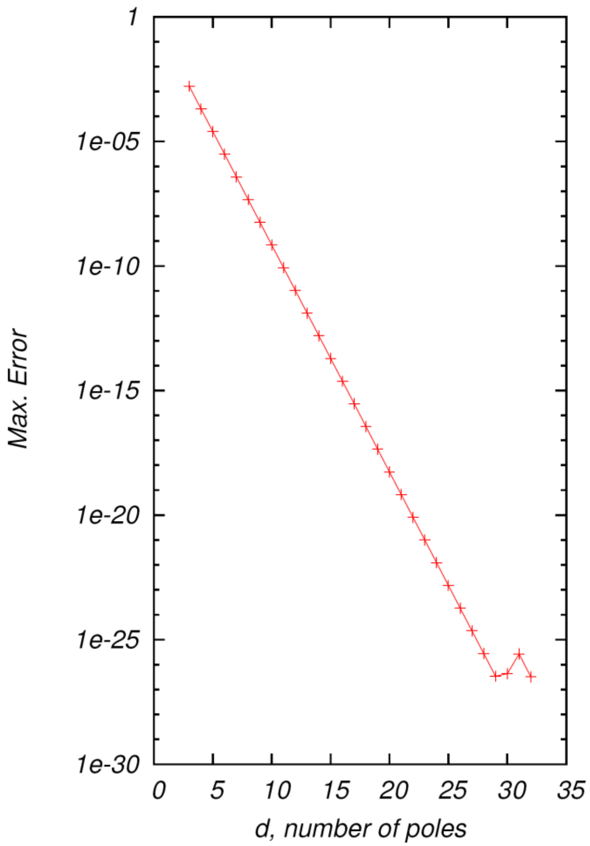
Pade approximant reduces to purely imaginary line
(use Alpert-Greengard-Hagstrom algorithm)

$$z(\zeta) = z_0 + \sum_{k=1}^{N} \frac{\gamma_k}{\zeta - i\,|\beta_k|}$$

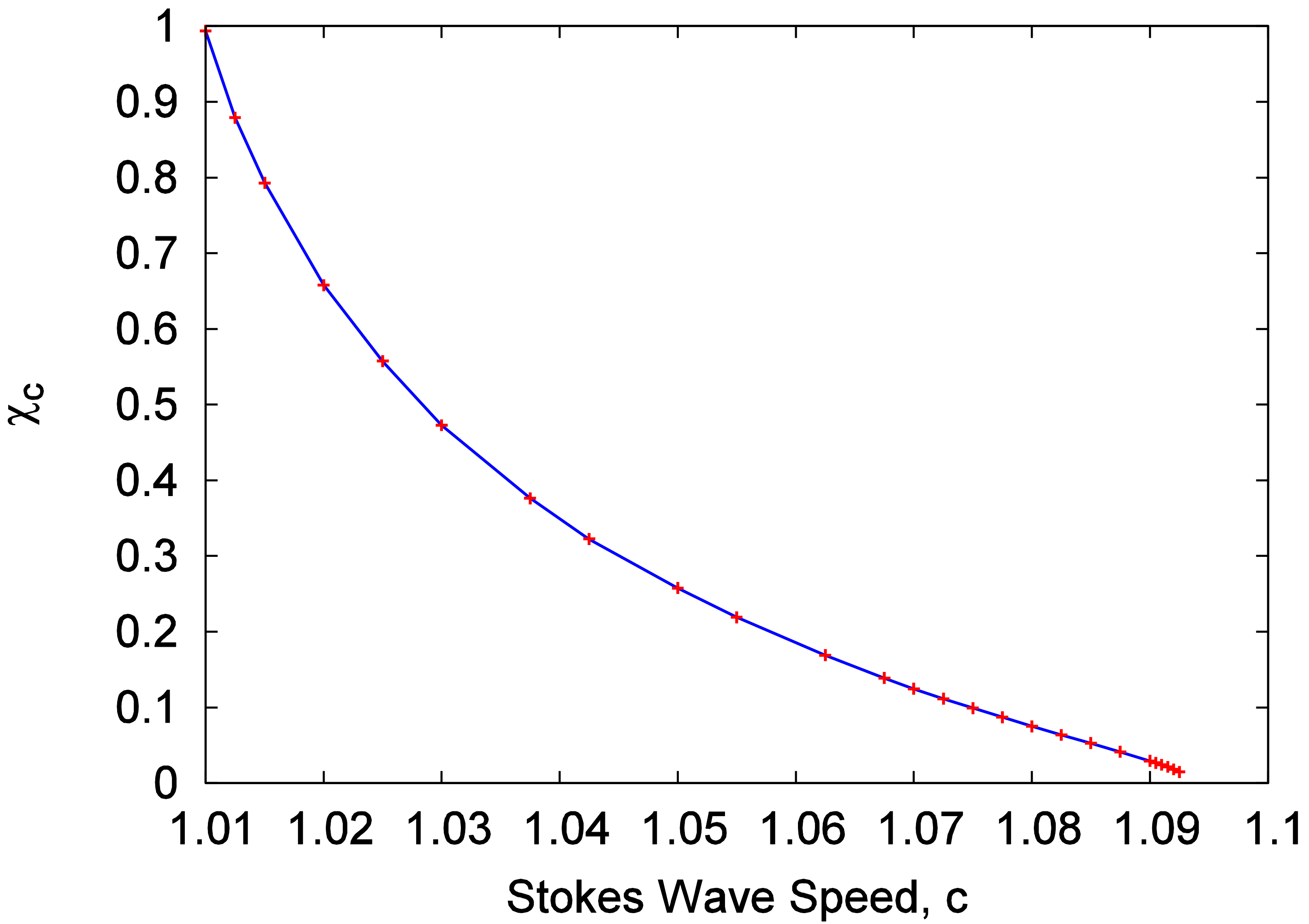
Position of poles:



Error compare with Stokes wave:



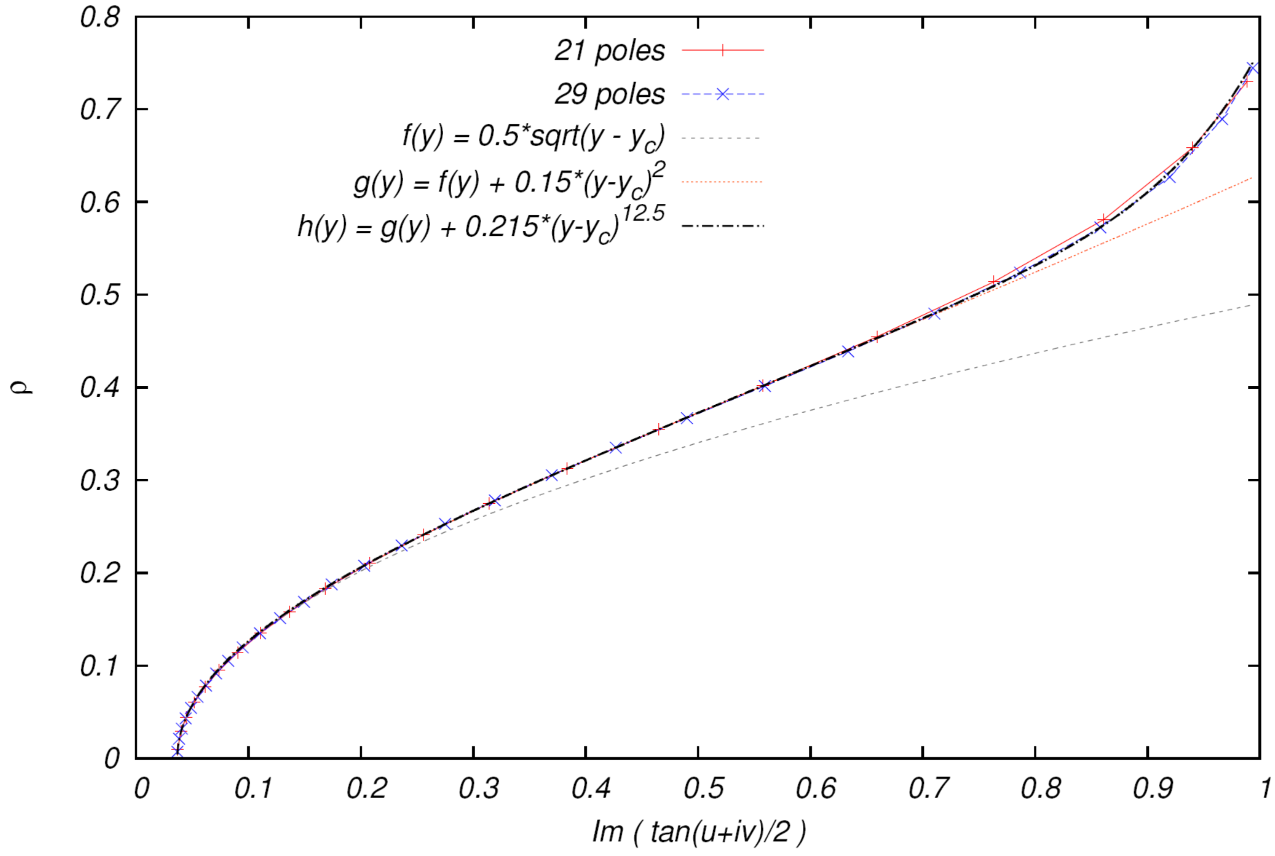
Location of closest singularity vs. Stokes wave velocity



Continuous limit: integral over branch cut

$$\tilde{z}(\zeta) = \int_{\chi_c}^{1} \frac{\rho(\chi')d\chi'}{\zeta - i\chi'} \approx \sum_{k=1}^{N} \frac{\gamma_k}{\zeta - i\,|\beta_k|}$$

$$\rho(\zeta) = \sum_{n=1}^{\infty} a_n(\zeta - i\chi_c)^{n/2} \qquad\qquad \zeta = \kappa + i\chi = \tan(\tfrac{w}{2})$$



Stokes wave equation

$$z_u = 1 + \frac{2g}{c^2}\hat{P}[y z_u]$$

$$\bar{\tilde{z}}\tilde{z}_u = -\frac{1}{2}(1+\zeta^2)\int_{\chi_c}^{1}\int_{\chi_c}^{1}\frac{\rho(\chi)\rho(\chi')d\chi d\chi'}{(\zeta+i\chi)(\zeta-i\chi')^2}$$

- easy to apply projector $\hat{P}$

$$y = -\frac{i}{2}(\tilde{z} - \bar{\tilde{z}})$$

$$\rho(\zeta) = \sum_{n=1}^{\infty} a_n(\zeta - i\chi_c)^{n/2}$$

$\Rightarrow$ Closed equations for amplitudes a_n

Conclusion

- Analytical properties of Stokes wave are fully determined by a single branch cut

-Solution for stokes waves is reduced to the evaluation of integrals along that branch cut

- That solution can be effectively found by the power series for the integral form

$$\tilde{z}(\zeta) = \int_{\chi_c}^{1} \frac{\rho(\chi')d\chi'}{\zeta - i\chi'}$$

$$\rho(\zeta) = \sum_{n=1}^{\infty} a_n(\zeta - i\chi_c)^{n/2}$$

$$\zeta = \kappa + i\chi = \tan(\tfrac{w}{2})$$