The two-dimensional 4-state Potts model in a magnetic field

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$$f_s(\tau, h) = b^{-D} F_\pm(\kappa_\tau b^{y_\tau} |\tau|, \kappa_h b^{y_h} |h|)$$

where D is the space dimension, b is the rescaling factor, τ and h are the relevant thermal and magnetic fields with corresponding RG eigenvalues y_τ and y_h, $F_\pm(x, y)$ is a universal function of its arguments x and y, $\pm$ stands for $T > T_c$ and $T < T_c$, and κ_τ and κ_h are non universal metric factors (i.e. which depend e.g. on lattice symmetry at a given space dimension).

$$
\begin{aligned}
M_c(0,h) &= \kappa_h b^{-D+y_h} \partial_h F_-(x,y)\big|_{x=0}, & h \to 0 \\
M_-(\tau,0) &= \kappa_h b^{-D+y_h} \partial_h F_-(x,y)\big|_{y=0}, & \tau \to 0^-, \\
\chi_\pm(\tau,h) &= \kappa_h^2 b^{-D+2y_h} \partial_h^2 F_\pm(x,y), & \tau \to 0,\ h \to 0 \\
C_\pm(\tau,h) &= \kappa_\tau^2 b^{-D+2y_\tau} \partial_\tau^2 F_\pm(x,y), & \tau \to 0,\ h \to 0.
\end{aligned}
$$

$$b = (\kappa_h |h|)^{-1/y_h}, \tau = 0,\ h \to 0$$

$$M_c(h) = D_c^{-1/\delta} |h|^{1/\delta}, \quad \delta = \frac{y_h}{D-y_h}, \qquad D_c^{-1/\delta} = \kappa_h^{1+1/\delta} \partial_h F_c(0,1)$$

$$b = (\kappa_\tau |\tau|)^{-1/y_\tau}, \tau \to 0,\ h = 0$$

$$
\begin{aligned}
M_-(\tau) &= B_- |\tau|^{\beta}, & \beta &= \frac{D-y_h}{y_\tau}, & B_- &= \kappa_h \kappa_\tau^{\beta} \partial_h F_-(1,0), \\
\chi_\pm(\tau) &= \Gamma_\pm |\tau|^{-\gamma}, & \gamma &= \frac{2y_h-D}{y_\tau}, & \Gamma_\pm &= \kappa_h^2 \kappa_\tau^{-\gamma} \partial_h^2 F_\pm(1,0), \\
C_\pm(\tau,h) &= \frac{A_\pm}{\alpha} |\tau|^{-\alpha}, & \alpha &= \frac{2y_\tau-D}{y_\tau}, & \frac{A_\pm}{\alpha} &= \kappa_\tau^{2-\alpha} \partial_\tau^2 F_\pm(1,0).
\end{aligned}
$$

$$M_-(\tau,0) = \kappa_h b^{-D+y_h}\partial_h F_-(x,y)|_{y=0}, \quad \tau\to 0^-,$$
$$\chi_\pm(\tau,h) = \kappa_h^2 b^{-D+2y_h}\partial_h^2 F_\pm(x,y), \quad \tau\to 0,\ h\to 0$$
$$C_\pm(\tau,h) = \kappa_\tau^2 b^{-D+2y_\tau}\partial_\tau^2 F_\pm(x,y), \quad \tau\to 0,\ h\to 0$$

the function $R_C^\pm(\tau) = C_\pm(\tau)\chi_\pm(\tau)|\tau|^2/M_-^2(\tau)$

tends to the universal quantity

$$R_C^\pm = \partial_\tau^2 F_\pm(1,0)\partial_h^2 F_\pm(1,0)/(\partial_h F_\pm(\bar{1},0))^2$$

$$M_-(\tau) = B_-|\tau|^\beta,$$
$$\chi_\pm(\tau) = \Gamma_\pm|\tau|^{-\gamma},$$
$$C_\pm(\tau,h) = \frac{A_\pm}{\alpha}|\tau|^{-\alpha}$$

$$R_C^\pm = A_\pm\Gamma_\pm/\alpha B_-^2$$

$$2\beta+\gamma = 2-\alpha$$

logarithmic corrections involve "hat exponents"

$$M_-(\tau) = B_-|\tau|^{\beta}(-\ln|\tau|)^{\hat{\beta}}$$

$$\chi_\pm(\tau) = \Gamma_\pm|\tau|^{-\gamma}(-\ln|\tau|)^{\hat{\gamma}}$$

$$C_\pm(\tau) = \frac{A_\pm}{\alpha}|\tau|^{-\alpha}(-\ln|\tau|)^{\hat{\alpha}}$$

$$R_C^\pm(\tau) \to \frac{A_\pm\Gamma_\pm}{\alpha B_-^2}(-\ln|\tau|)^{\hat{\alpha}-2\hat{\beta}+\hat{\gamma}}$$

$$\hat{\alpha}-2\hat{\beta}+\hat{\gamma} = 0$$

$$R_\chi^\pm = \Gamma_\pm D_c B_-^{\delta-1} \qquad \beta(\delta-1)=\gamma$$

$$R_\chi(\tau,h) = \chi_\pm(\tau) M_-^{\delta-1}(\tau) M_c^{-\delta}(h)|h|$$

$$\partial_h^2 F_\pm(1,0)(\partial_h F_\pm(1,0))^{\delta-1}(\partial_h F_c(0,1))^{-\delta}$$

When logarithmic corrections

$$R_\chi(\tau,h) \to \Gamma_\pm D_c B_-^{\delta-1}(-\ln|\tau|)^{\hat\gamma+\hat\beta(\delta-1)}(-\ln|h|)^{-\delta\hat\delta}$$

$$\hat\gamma + \hat\beta(\delta-1) = 0 \text{ and } \delta\hat\delta = 0$$

$$\frac{d\tau}{dl} = (y_\tau + y_{\tau\psi}\psi)\tau,$$
$$\frac{dh}{dl} = (y_h + y_{h\psi}\psi)h,$$
$$\frac{d\psi}{dl} = g(\psi).$$

$$g(\psi) = y_{\psi^2}\psi^2 + y_{\psi^3}\psi^3 + \ldots.$$

with $l = \ln b$. The fixed point is at $\tau = h = 0$.

$y_{\tau\psi} = 3/(4\pi)$, $y_{h\psi} = 1/(16\pi)$, $y_{\psi^2} = 1/\pi$

$y_{\psi^3} = -1/(2\pi^2)$ $\quad$ $y_\tau = 3/2$ and $y_h = 15/8$.

$$\frac{d\tau}{dl} = (y_\tau + y_{\tau\psi}\psi)\tau,$$
$$\frac{dh}{dl} = (y_h + y_{h\psi}\psi)h,$$
$$\frac{d\psi}{dl} = g(\psi).$$

$$\begin{aligned} f_s(\tau(0), 0, \psi(0)) &= e^{-Dl} f_s(\tau(l), 0, \psi(l)), \\ \xi(\tau(0), 0, \psi(0)) &= e^{l} \xi(\tau(l), 0, \psi(l)), \end{aligned}$$

$$f_s(0, h(0), \psi(0)) = e^{-Dl} f_s(0, h(l), \psi(l))$$

$$\frac{d\tau}{dl} = (y_\tau + y_{\tau\psi}\psi)\tau,$$
$$\frac{dh}{dl} = (y_h + y_{h\psi}\psi)h,$$
$$\frac{d\psi}{dl} = g(\psi). \qquad g(\psi) = y_{\psi^2}\psi^2 + y_{\psi^3}\psi^3 + \ldots .$$

$$\int_0^l \frac{d\varphi}{\varphi} = \ln\frac{\varphi(l)}{\varphi(0)} = \mathrm{const} + \ln\frac{1}{|\varphi|} = y_\varphi l + y_{\varphi\psi}\int_0^l \psi dl$$

$$\psi dl = \frac{1}{y_{\psi^2}}\left(\frac{1}{\psi} - \frac{y_{\psi^3}}{y_{\psi^2} + y_{\psi^3}\psi}\right) d\psi$$

$$\int_0^l \psi dl = \frac{1}{y_{\psi^2}} \ln\left(\frac{\psi(l)}{\psi(0)}\frac{y_{\psi^2} + y_{\psi^3}\psi(0)}{y_{\psi^2} + y_{\psi^3}\psi(l)}\right)$$

$$l = \mathrm{const} - \frac{1}{y_\varphi}\ln|\varphi| + \frac{y_{\varphi\psi}}{y_\varphi y_{\psi^2}} \ln\frac{\psi(0)}{\psi(l)}\frac{y_{\psi^2} + y_{\psi^3}\psi(l)}{y_{\psi^2} + y_{\psi^3}\psi(0)}$$

$$\frac{d\tau}{dl} = (y_\tau + y_{\tau\psi}\psi)\tau,$$
$$\frac{dh}{dl} = (y_h + y_{h\psi}\psi)h,$$
$$\frac{d\psi}{dl} = g(\psi).$$

$$g(\psi) = y_{\psi^2}\psi^2 + y_{\psi^3}\psi^3 + \ldots.$$

$$f_s(\tau(0), 0, \psi(0)) = e^{-Dl} f_s(\tau(l), 0, \psi(l)),$$
$$\xi(\tau(0), 0, \psi(0)) = e^{l}\xi(\tau(l), 0, \psi(l)),$$

$$l = \mathrm{const} - \frac{1}{y_\varphi}\ln|\varphi| + \frac{y_{\varphi\psi}}{y_\varphi y_{\psi^2}}\ln\frac{\psi(0)}{\psi(l)}\frac{y_{\psi^2} + y_{\psi^3}\psi(l)}{y_{\psi^2} + y_{\psi^3}\psi(0)}$$

$$\zeta = \frac{\psi(l)}{\psi(0)}\frac{y_{\psi^2} + y_{\psi^3}\psi(0)}{y_{\psi^2} + y_{\psi^3}\psi(l)}$$

$$
\begin{aligned}
f_s(\varphi, \psi(0)) &= \text{const} \times |\varphi|^{D/y_\varphi} \zeta^{D y_{\varphi\psi}/y_\varphi y_{\psi^2}} \\
\xi(\varphi, \psi(0)) &= \text{const} \times |\varphi|^{-1/y_\varphi} \zeta^{-y_{\varphi\psi}/y_\varphi y_{\psi^2}}
\end{aligned}
$$

$$f_s(\tau, \psi) \sim |\tau|^{D\nu} \zeta^{D\mu}$$

$$f_s(h, \psi) \sim |h|^{D\nu_c} \zeta^{D\mu_c}$$

$$\zeta \sim \frac{1}{-\ln|\varphi|}(1 + \text{corrections})$$

$$h = 0,\ \tau \to 0^{\pm}, \qquad \tau = 0,\ h \to 0^{\pm},$$

$$f_s(\tau,\psi) = F_{\pm}|\tau|^{2-\alpha}(-\ln|\tau|)^{\hat{\alpha}}, \qquad f_s(h,\psi) = F_c|h|^{1+1/\delta}(-\ln|h|)^{\hat{\delta}},$$

$$M(\tau,\psi) = B_{-}|\tau|^{\beta}(-\ln|\tau|)^{\hat{\beta}},\ (\tau<0), \qquad M(h,\psi) = D_c^{-1/\delta}|h|^{1/\delta}(-\ln|h|)^{\hat{\delta}},$$

$$E(\tau,\psi) = \frac{A_{\pm}}{\alpha(1-\alpha)}|\tau|^{1-\alpha}(-\ln|\tau|)^{\hat{\alpha}}, \qquad E(h,\psi) = E_c|h|^{\epsilon_c}(-\ln|h|)^{\hat{\epsilon}_c},$$

$$\chi(\tau,\psi) = \Gamma_{\pm}|\tau|^{-\gamma}(-\ln|\tau|)^{\hat{\gamma}}, \qquad \chi(h,\psi) = \Gamma_c|h|^{1/\delta-1}(-\ln|h|)^{\hat{\delta}},$$

$$C(\tau,\psi) = \frac{A_{\pm}}{\alpha}|\tau|^{-\alpha}(-\ln|\tau|)^{\hat{\alpha}}, \qquad C(h,\psi) = \frac{A_c}{\alpha_c}|h|^{-\alpha_c}(-\ln|h|)^{\hat{\alpha}_c},$$

$$\xi(\tau,\psi) = \xi_{\pm}|\tau|^{-\nu}(-\ln|\tau|)^{\hat{\nu}}, \qquad \xi(h,\psi) = \xi_c|h|^{-\nu_c}(-\ln|h|)^{\hat{\nu}_c}.$$

$$\alpha = 2 - D\nu = \frac{2}{3}, \qquad \alpha_c = 2\frac{\nu_c}{\nu} - \frac{D}{\nu_c} = \frac{8}{15},$$

$$\beta = D\nu - \frac{\nu}{\nu_c} = \frac{1}{12}, \qquad \delta = \frac{1}{D\nu_c - 1} = 15,$$

$$\gamma = 2\frac{\nu}{\nu_c} - D\nu = \frac{7}{6}, \qquad \epsilon_c = D\nu_c - \frac{\nu_c}{\nu} = \frac{4}{15},$$

$$\nu = \frac{2}{3}, \qquad \nu_c = \frac{8}{15}.$$

$$\hat{\alpha} = -D\mu = -1, \qquad \hat{\alpha}_c = 2\frac{\mu_c - \mu)}{\nu} - D\nu_c = -\frac{22}{15},$$

$$\hat{\beta} = \frac{\mu - \mu_c}{\nu_c} - D\mu = -\frac{1}{8}, \qquad \hat{\delta} = -D\mu_c = -\frac{1}{15},$$

$$\hat{\gamma} = 2\frac{\mu - \mu_c}{\nu_c} - D\mu = \frac{3}{4}, \qquad \hat{\epsilon}_c = \frac{\mu_c - \mu}{\nu} - D\mu_c = -\frac{23}{30},$$

$$\hat{\nu} = \mu = \frac{1}{2}, \qquad \hat{\nu}_c = \mu_c = \frac{1}{30}.$$

$$R_\chi^\pm(\tau,h) = |h|\chi_\pm(\tau,\zeta)M_-^{\delta-1}(\tau,\zeta)M_c^{-\delta}(h,\zeta)$$

$$\hat{\gamma} + (\delta-1)\hat{\beta} - \delta\hat{\delta} = 0$$