Intstanton blow up equations and CFT

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26 June 2013

The main equations

- Let $b_1 = b/\sqrt{1-b^2}$, $b_2 = \sqrt{b^2-1}$, b is generic.
  Let $P \notin \{P_{m,n}\}$, $P_1 = \sqrt{b^{-1}/(b^{-1}-b)}P$, $P_2 = \sqrt{b/(b-b^{-1})}P$.

$$\mathbb{F}(P,b;q) = \sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{l_k^2(P,b)} \cdot \mathbb{F}\left(P_1+kb_1, b_1; \beta_1 q\right) \cdot \mathbb{F}\left(P_2+kb_2^{-1}, b_2; \beta_2 q\right), \qquad (1)$$

where $\beta_1 = \dfrac{b^{-2}}{(b^{-1}-b)^2}$, $\beta_2 = \dfrac{b^2}{(b-b^{-1})^2}$,

- Nakajima Yoshioka (2003)

$$Z(\epsilon_1,\epsilon_2,a;q) = \sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{l_k} Z(\epsilon_1,\epsilon_2-\epsilon_1,a+k\epsilon_1;q)\cdot Z(\epsilon_1-\epsilon_2,\epsilon_2,a+k\epsilon_2;q), \quad (2)$$

where $Z(\epsilon_1,\epsilon_2,a;q)$ — Nekrasov partition function

- Alday, Gaiotto, Tachikawa (2009) $b=\sqrt{\epsilon_1/\epsilon_2}$, $P = a/\sqrt{\epsilon_1\epsilon_2}$

$$F\left(\sqrt{\frac{\epsilon_1}{\epsilon_2}}, \frac{a}{\sqrt{\epsilon_1\epsilon_2}}, (\epsilon_1\epsilon_2)^2 q\right) = Z(\epsilon_1,\epsilon_2,a;q), \qquad (3)$$

Virasoro algebra

- By Vir we denote the Virasoro Lie algebra with the generators $L_n, n \in \mathbb{Z}, C$ subject of relation:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{n^3-n}{12}C, \quad [L_n, C] = 0$$

- Denote by $\mathrm{V}_{\Delta,c}$ the Verma module of the Virasoro algebra generated by the highest weight vector v:

$$L_n v = 0, \text{ for } n > 0 \qquad L_0 v = \Delta v, \ Cv = cv.$$

- By $\mathbb{L}_{\Delta,c}$ denote its irreducible quotient.
- It is convenient to parametrize Δ and c as

$$\Delta = \Delta(P,b) = \frac{(b^{-1}+b)^2}{4} - P^2, \qquad c = 1 + 6(b^{-1}+b)^2$$

We denote the corresponding irreducible representation as $\mathbb{L}_{\mathrm{P},b}$

Whittaker limit of conformal block

- The Whittaker vector $\mathrm{W} = \sum_{N=0} w_N q^{N/2}$, defined by the equations:

$$L_0 w_N = (\Delta + N) w_N, \quad L_1 w_N = w_{N-1}, \quad L_k w_N = 0, \text{for } k > 1.$$

  These equations can be simply rewritten as $L_1 \mathrm{W} = q^{1/2}\mathrm{W}$, $L_k \mathrm{W} = 0$, for $k > 1$.
- We will always use normalisation of W such that $\langle w_0, w_0 \rangle = 1$. Therefore

$$w_0 = v, \qquad w_1 = \frac{1}{2\Delta} L_{-1} v$$

$$w_2 = \frac{c + 8\Delta}{4\Delta(c - 10\Delta + 2c\Delta + 10\Delta^2)} L_{-1}^2 v - \frac{3}{c - 10\Delta + 2c\Delta + 10\Delta^2} L_{-2} v$$

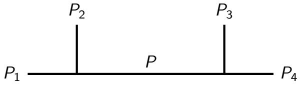
- The Whittaker vector corresponding to $\mathrm{V}_{\mathrm{P},b}$ will be denoted by $\mathrm{W}_{\mathrm{P},b}$.
- The Whittaker limit of the 4 point conformal block defined by:

$$\mathbb{F}(P, b; q) = \langle \mathrm{W}_{\mathrm{P},b}, \mathrm{W}_{\mathrm{P},b} \rangle = \sum_{N=0}^{\infty} \langle w_{\mathrm{P},b,N}, w_{\mathrm{P},b,N} \rangle q^N \tag{4}$$

$$\mathbb{F}(P, b; q) = 1 + \frac{2}{(b + b^{-1})^2 - 4P^2} q + \dots$$

4 point conformal block

- Correlation functions in Conformal Field Theory is product of structure constants, holomorphic and antiholomorphic functions. This holomorphic function is called *conformal block*.
- The first nontrivial example is a 4 point conformal block $\mathbb{F}(P_1, P_2, P_3, P_4, P, b; q)$



The Whittaker limit $\mathbb{F}(P, b; q)$ defined by $P_2, P_3 \to \infty$.

- [Dotsenko,Fateev] if $P_2 = b^{-1} + nb$, $P = P_1 + nb$, the function $\mathbb{F}$ is given by n-tuple contour integral (for $n = 2$ the $\mathbb{F}$ is a ${}_2F_1$.):

$$\mathbb{F}(P_1, P_2, P_3, P_4, P, b; q) = \int dt_1 \ldots \int dt_{n-1} \prod_i t_i^a (t_i - 1)^b (t_i - q)^c \prod_{i,j} (t_i - t_j)^g$$

- If $b^2 = p/p'$, $\mathcal{M}(p, p')$ Minimal Model, in particular $\mathcal{M}(3, 4)$ — Ising.
- [Gamayun, Iorgov, Lisovyy] For $c = 1$ we have τ-function of Painleve VI

Nekrasov partition function

- Denote by $M(r,N)$ the moduli space of framed torsion free sheaves on $\mathbb{CP}^2$ of rank r, $c_1 = 0$, $c_2 = N$.
- This space in a smooth partial compactification of *moduli space of* $U(r)$ *instantons*: $F = -*F$
- $M(r,N)$ is smooth manifold of complex dimension $2rN$.
- There is a natural action of the $r+2$ dimensional torus T on the $M(r,N)$: $(\mathbb{C}^*)^2$ acts on the base $\mathbb{CP}^2$ and $(\mathbb{C}^*)^r$ acts on the framing ant the infinity.
- The Nekrasov partition function for pure Yang-Mills theory is defined as the equivariant volume:

$$Z(\epsilon_1,\epsilon_2,\vec{a};q) = \sum_{N=0}^{\infty} q^N \int_{M(r,N)} [1],$$

where $\vec{a} = (a_1,\ldots,a_r)$ and $\epsilon_1,\epsilon_2,a_1,\ldots,a_r$ are the coordinates on the $\mathfrak{t} = \mathrm{Lie}\,T$.

- The last integrals can be computed by localisation method and equal to the sum of contributions of torus fixed points (which are labeled by r-tuple of Young diagrams $\lambda_1\ldots,\lambda_r$).

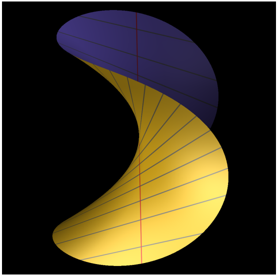
Blow up

- Blow up

$$\widehat{\mathbb{C}^2} = \Big\{ z_1, z_2, u, v \big| z_1 v = z_2 u, (u,v) \neq (0,0) \Big\} \Big/ \mathrm{GL}(1),$$

where $\mathrm{GL}(1)$ acts by $(z_1, z_2, u, v) \to (z_1, z_2, tu, tv)$

- The projection map $\widehat{\pi} \colon \widehat{\mathbb{C}^2} \to \mathbb{C}^2$, $(z_1, z_2, u, v) \mapsto (z_1, z_2)$



Blow up equations

- Denote by $\widehat{M}(r,k,N)$ moduli space framed torsion free shaeves on $\widehat{\mathbb{C}^2}$, $\mathrm{rk}(\mathcal{E})=r$, $c_1(\mathcal{E})=k$, $c_2(\mathcal{E})-\frac{r-1}{2r}c_1(\mathcal{E})^2=N$.

$$\widehat{Z}(\epsilon_1,\epsilon_2,\vec{a};q)=\sum_{N=0}^{\infty}q^N\int_{\widehat{M}(r,0,N)}[1],$$

- The manifolds $\widehat{M}(r,0,N)$ is nonsingular of dimension $2rN$
- There is a map $\widehat{\pi}\colon\widehat{M}(r,0,N)\to M_0(r,N)$

$$\widehat{Z}(\epsilon_1,\epsilon_2,\vec{a};q)=Z(\epsilon_1,\epsilon_2,\vec{a};q)$$

- There are two torus invariant points on the $\widehat{\mathbb{C}^2}$: $(0,0,1,0)$ and $(0,0,0,1)$.
- The torus fixed points on the $\widehat{M}(r,0,N)$ are labelled by $\vec{\lambda}^1,\vec{\lambda}^2,k$, such that $N=|\vec{\lambda}|+|\vec{\lambda}|+k^2$.

$$\widehat{Z}(\epsilon_1,\epsilon_2,a;q)=\sum_{k\in\mathbb{Z}}\frac{q^{k^2}}{l_k}Z(\epsilon_1,\epsilon_2-\epsilon_1,a+k\epsilon_1;q)\cdot Z(\epsilon_1-\epsilon_2,\epsilon_2,a+k\epsilon_2;q),$$

Main Theorem

- Recall that for generic b

$$\mathbb{F}(P,b;q)=\sum_{k\in\mathbb{Z}}\frac{q^{k^2}}{l_k^2(P,b)}\cdot\mathbb{F}\left(P_1+kb_1,b_1;\beta_1 q\right)\cdot\mathbb{F}\left(P_2+kb_2^{-1},b_2;\beta_2 q\right),$$

where $b_1=b/\sqrt{1-b^2}$, $b_2=\sqrt{b^2-1}$ $\beta_1=\dfrac{b^{-2}}{(b^{-1}-b)^2}$, $\beta_2=\dfrac{b^2}{(b-b^{-1})^2}$,

- Denote by $\mathcal{M}_b$ the Vertex operator algebra Vir with the central charge $c=1+6(b^{-1}+b)^2$.
- Denote by $\mathcal{A}_b$ the product $\mathcal{M}_{b_1}\otimes\mathcal{M}_{b_2}$ extended by the fields $\Phi_{1,2}^{b_1}\otimes\Phi_{2,1}^{b_2}$

$$\mathcal{A}_b=\left(\mathbb{L}_{(1,1)}^{b_1}\otimes\mathbb{L}_{(1,1)}^{b_2}\right)\oplus\left(\mathbb{L}_{(1,3)}^{b_1}\otimes\mathbb{L}_{(3,1)}^{b_2}\right)\oplus\left(\mathbb{L}_{(1,5)}^{b_1}\otimes\mathbb{L}_{(5,1)}^{b_2}\right)\oplus\dots$$

Theorem

There is an isomorphism of algebras $\mathcal{A}_b\cong\mathcal{M}_b\otimes\mathcal{U}$

Field $\varphi(z)$

- Let φ_n be a generators of the Heisenberg algebra: $[\varphi_n, \varphi_m] = n\delta_{m+n,0}$. It is convenient to consider operators φ_n as modes of the bosonic field $\varphi(z)$:

$$\varphi(z) = \sum_{n\in\mathbb{Z}\setminus 0} \frac{\varphi_n}{-n} z^{-n} + \varphi_0 \log z + \widehat{Q},$$

where the operator $\widehat{Q}$ is conjugate to the operator $\widehat{P} = \varphi_0$, i.e. satisfy the relation: $[\widehat{P}, \widehat{Q}] = 1$. The relation of the Heisenberg algebra can be rewritten in terms of operator product expansion:

$$\varphi(z)\varphi(w) = \log(z - w) + \mathrm{reg}.$$

- Denote by F_λ the Fock representation of the Heisenberg algebra with the highest weight vector v_λ:

$$\varphi_n v_\lambda = 0 \ \text{for}\ n > 0, \quad \varphi_0 v_\lambda = \lambda v_\lambda.$$

Denote by S_λ the shift operator $S_\lambda \colon \mathrm{F}_\mu \to \mathrm{F}_{\mu+\lambda}$ defined by

$$S_\lambda v_\mu = v_{\mu+\lambda}, \qquad [S_\lambda, \varphi_n] = 0, \ \text{for}\ n \neq 0$$

Actually S_λ is just an exponent $\exp(\lambda\widehat{Q})$.

Lattice $\mathbb{Z}\sqrt{2}$

- The direct sum $V_{\sqrt{2}\mathbb{Z}} := \bigoplus_{k\in\mathbb{Z}} \mathrm{F}_{k\sqrt{2}}$ is a vacuum representation of the lattice vertex operator algebra.
- Under the operator-state correspondence the highest weight vectors v_λ, $\lambda = k\sqrt{2}$ correspond to

$$Y(v_\lambda; z) =: e^{\lambda\varphi} := S_\lambda z^{\lambda\varphi_0} \exp\Big(\lambda \sum_{n\in\mathbb{Z}_{>0}} \frac{\varphi_{-n}}{n} z^n\Big) \exp\Big(\lambda \sum_{n\in\mathbb{Z}_{>0}} \frac{\varphi_n}{-n} z^{-n}\Big),$$

  Here and below $: \ldots :$ denotes the creation-annihilation normal ordering.
- For the more general vectors of the form $v = \varphi_{-m}^{n_m} \cdots \varphi_{-1}^{n_1} v_\lambda$ the corresponding operator have the form:

$$Y(v; z) =: (\partial^m\varphi)^{n_m} \cdots (\partial\varphi)^{n_1} e^{\lambda\varphi} :$$

- The stress-energy tensor $T(z) = \frac{1}{2}(\partial\varphi)^2$ have central charge $c = 1$. More general $T(z) = \frac{1}{2}(\partial\varphi)^2 + u(\partial^2\varphi)$ satisfy stress-energy relation with the central charge $c = 1 - 12u^2$.

- $T(z)$ is called *stress-energy tensor*, parameter c is called the *central charge*

$$T(z)T(w)=\frac{c}{(z-w)^4}+\frac{2}{(z-w)^2}T(w)+\frac{1}{(z-w)}\partial T(w)+\mathrm{reg.} \qquad (5)$$

Definition

The conformal algebra $\mathcal{U}$ coincide with the $V_{\sqrt{2}\mathbb{Z}}$ as the operator algebra, but the stress–energy tensor is modified:

$$T_{\mathcal{U}}=\frac{1}{2}(\partial\varphi)^2+\frac{1}{\sqrt{2}}(\partial^2\varphi)+\epsilon\left(2(\partial\varphi)^2e^{\sqrt{2}\varphi}+\sqrt{2}(\partial^2\varphi)e^{\sqrt{2}\varphi}\right)=$$
$$=\frac{1}{2}\partial_z\varphi(z)^2+\frac{1}{\sqrt{2}}\partial_z^2\varphi(z)+\epsilon\partial_z^2(e^{\sqrt{2}\varphi(z)}),\quad \varepsilon\neq 0 \qquad (6)$$

- The conformal algebras $\mathcal{U}$ isomorphic for different values $\varepsilon\neq 0$. For the $\varepsilon=0$ $T_{\mathcal{U}}(z)$ has the from discussed above form for $u=\frac{1}{\sqrt{2}}$ and central charge -5.
- The spaces $U_0=\bigoplus_{k\in\mathbb{Z}}\mathrm{F}_{k\sqrt{2}}$ and $U_1=\bigoplus_{k\in\mathbb{Z}+1/2}\mathrm{F}_{k\sqrt{2}}$ become a representations of $\mathcal{U}$.

Picture of $\mathcal{U}$

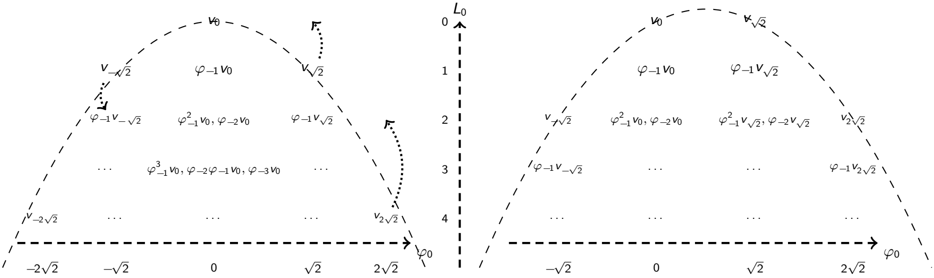


Figure: The basic vectors with the lowest L_0 grading. The left part correspond to the vacuum representation of $V_{\sqrt{2}\mathbb{Z}}$, the right part correspond to the vacuum representaion of $\mathcal{U}$. Dotted curved arrows shows shift of L_0 grading to $L_0 - \frac{1}{\sqrt{2}}\varphi_0$.

Minimal midels

- Let $b^2 = -p/p'$. By $\mathcal{A}_{p/p'}$ we denote an extension of the product of minimal models $\mathcal{M}_{p/(p+p')} \otimes \mathcal{M}_{(p+p')/p'}$ by the field $\Phi_{(1,2)})] \cdot \Phi_{(2,1)}$.

Conjecture (Theorem)

There is an isomorphism of algebras $\mathcal{A}_{p/p'} \cong \mathcal{M}_{p/p'} \otimes \mathcal{U}$

- Let $(p, p') = (2, 3)$. Then $c_{2/3} = 0$, and the $\mathcal{M}_{2/3}$ is an empty theory. Therefore $\mathcal{U} = \mathcal{A}_{2/3} \supset \mathcal{M}_{2/5} \otimes \mathcal{M}_{5/3}$

$$T_{2/5} = -\frac{1}{10\epsilon}e^{-\sqrt{2}\varphi}+\frac{1}{5}(\partial\varphi)^2+\frac{3}{5\sqrt{2}}(\partial^2\varphi)+\frac{12\epsilon}{5}(\partial\varphi)^2e^{\sqrt{2}\varphi}+\frac{3\sqrt{2}\epsilon}{5}(\partial^2\varphi)e^{\sqrt{2}\varphi}-\frac{12\epsilon^2}{5}e^{2\sqrt{2}\varphi},$$

$$T_{5/3} = \frac{1}{10\epsilon}e^{-\sqrt{2}\varphi}+\frac{3}{10}(\partial\varphi)^2+\frac{2}{5\sqrt{2}}(\partial^2\varphi)-\frac{2\epsilon}{5}(\partial\varphi)^2e^{\sqrt{2}\varphi}+\frac{2\sqrt{2}\epsilon}{5}(\partial^2\varphi)e^{\sqrt{2}\varphi}+\frac{12\epsilon^2}{5}e^{2\sqrt{2}\varphi}.$$

Direct calculation shows that $T_{2/5}$ and $T_{5/3}$ commute and satisfy (5) with the central charges $c_{2/5} = -\frac{22}{5}$ and $c_{5/3} = -\frac{3}{5}$ correspondingly. It is clear that

$$T_{\mathcal{U}} = T_{2/5} + T_{5/3}.$$

$\mathcal{A}_b = \mathcal{U} \otimes \mathcal{M}_b$

- Recall that $\mathcal{A}_b$ is an extension of the product $\mathcal{M}_{b_1} \otimes \mathcal{M}_{b_2}$.
- One can find this to commution Virasoro algebras in the product $\mathcal{U} \otimes \mathcal{M}_b$

$$
\begin{aligned}
T_{b_1} = \frac{b+b^{-1}}{2(b-b^{-1})\epsilon} e^{-\sqrt{2}\varphi} + \frac{b}{2(b-b^{-1})}(\partial\varphi)^2 - \frac{b^{-1}}{\sqrt{2}(b-b^{-1})}\partial^2\varphi - \\
- \frac{(1+2b^{-2})\epsilon}{b^2-b^{-2}}(\partial\varphi)^2 e^{\sqrt{2}\varphi} - \frac{\sqrt{2}b^{-1}\epsilon}{b-b^{-1}}(\partial^2\varphi)e^{\sqrt{2}\varphi} - \frac{2\epsilon^2}{b^2-b^{-2}}e^{2\sqrt{2}\varphi} - \\
- \frac{b^{-1}}{b-b^{-1}}T_b - \frac{2\epsilon}{b^2-b^{-2}}T_b e^{\sqrt{2}\varphi}
\end{aligned}
$$

$$
\begin{aligned}
T_{b_2} = -\frac{b+b^{-1}}{2(b-b^{-1})\epsilon} e^{-\sqrt{2}\varphi} - \frac{b^{-1}}{2(b-b^{-1})}(\partial\varphi)^2 + \frac{b}{\sqrt{2}(b-b^{-1})}\partial^2\varphi + \\
+ \frac{(2b^2+1)\epsilon}{b^2-b^{-2}}(\partial\varphi)^2 e^{\sqrt{2}\varphi} + \frac{\sqrt{2}b\epsilon}{b-b^{-1}}(\partial^2\varphi)e^{\sqrt{2}\varphi} + \frac{2\epsilon^2}{b^2-b^{-2}}e^{2\sqrt{2}\varphi} + \\
+ \frac{b}{b-b^{-1}}T_b + \frac{2\epsilon}{b^2-b^{-2}}T_b e^{\sqrt{2}\varphi}
\end{aligned}
$$

- T_{b_1} and T_{b_2} commute and sutisfy (5). The sum $T_{b_1} + T_{b_2} = T_{\mathcal{U}} + T_b$.

Representations of $\mathcal{A}_b = U \otimes \mathcal{M}_b$

- Let $P \notin \{P_{m,n}\}$, $P_1 = \sqrt{b^{-1}/(b^{-1} - b)}P$, $P_2 = \sqrt{b/(b - b^{-1})}P$.

Theorem

There is an isomorphism of representations:

$$U_1 \otimes \mathbb{L}_{\mathrm{P},b} = \bigoplus_{k\in\mathbb{Z}} \mathbb{L}_{(\mathrm{P}_1+kb_1),b_1} \otimes \mathbb{L}_{\left(\mathrm{P}_2+kb_2^{-1}\right),b_2},$$

$$U_0 \otimes \mathbb{L}_{\mathrm{P},b} = \bigoplus_{k\in\mathbb{Z}+\frac{1}{2}} \mathbb{L}_{(\mathrm{P}_1+kb_1),b_1} \otimes \mathbb{L}_{\left(\mathrm{P}_2+kb_2^{-1}\right),b_2}.$$

- The Whittaker vector

$$v_{\sqrt{1/2}} \otimes \mathrm{W}_{\mathrm{P},b}(q) = \sum_{k\in\mathbb{Z}} \frac{q^{k^2/2}}{l_k(P,b)} \left(\mathrm{W}_{P_1+kb_1,b_1}(\beta_1 q) \otimes \mathrm{W}_{P_2+kb_2^{-1},b_2}(\beta_2 q)\right),$$

where $\beta_1 = \frac{b^{-2}}{(b^{-1}-b)^2}$, $\beta_2 = \frac{b^2}{(b-b^{-1})^2}$

$$\mathbb{F}(P,b;q) = \sum_{k\in\mathbb{Z}} \frac{q^{k^2}}{l_k(P,b)^2} \cdot \mathbb{F}\left(P_1 + kb_1, b_1; \beta_1 q\right) \cdot \mathbb{F}\left(P_2 + kb_2^{-1}, b_2; \beta_2 q\right),$$

Differential equations

- Consider the operator $H = bL_0^{b_1} + b^{-1}L_0^{b_2}$. The corresponding local operator have the form:

$$bT_{b_1} + b^{-1}T_{b_2} = \frac{b+b^{-1}}{2\epsilon}e^{-\sqrt{2}\varphi} + \frac{b+b^{-1}}{2}(\partial\varphi)^2 + (b+b^{-1})\epsilon(\partial\varphi)^2 e^{\sqrt{2}\varphi} - \frac{2\epsilon^2}{b+b^{-1}}e^{2\sqrt{2}\varphi} - \frac{2\epsilon}{b+b^{-1}}T_b e^{\sqrt{2}\varphi} \qquad (7)$$

- Define the function $\widehat{\mathbb{F}}$ by:

$$\widehat{\mathbb{F}}(P,b;q,t) = \sum_{k=0}^{\infty}\widehat{\mathbb{F}}_m(P,b;q)\frac{t^m}{m!} = \left\langle v_{\sqrt{1/2}} \otimes \mathrm{W}_{\mathrm{P},b}, e^{tH}\left(v_{\sqrt{1/2}} \otimes \mathrm{W}_{\mathrm{P},b}\right)\right\rangle$$

- It is clear from the definition of operator H that:

$$\widehat{\mathbb{F}}(P,b;q,t) = \sum_{k\in\mathbb{Z}}\frac{q^{k^2}}{l_k(P,b)} \cdot e^{tb\Delta_k^1}\mathbb{F}\left(P+kb_1, b_1; \beta_1 q e^{tb}\right) \cdot e^{tb^{-1}\Delta_k^2}\mathbb{F}\left(P_2 + kb_2^{-1}, b_2; \beta_2 q e^{tb^{-1}}\right),$$

where $\Delta_k^1 = \Delta(P_1 + kb_1, b_1)$ and $\Delta_k^2 = \Delta(P_2 + kb_2^{-1}, b_2)$.

- We will use generalized Hirota-differential [?]:

$$\left(D_x^{(\epsilon_1,\epsilon_2)}\right)^m (f\cdot g) = (\frac{d}{dy})^m f(x+\epsilon_1 y)g(x+\epsilon_2 y)\Big|_{y=0}$$

- Therefore:

$$\widehat{\mathbb{F}}_m(P,b;q) = \sum_{k\in\mathbb{Z}} \frac{q^{1/4-\Delta(P,b)}}{l_k(P,b)} \left(D_{\log q}^{(b,b^{-1})}\right)^m \left(q^{\Delta_k^1}\mathbb{F}\left(P+kb_1,b_1;\beta_1 q\right)\cdot q^{\Delta_k^2}\mathbb{F}\left(P_2\right.\right.$$

where we used that $\Delta_k^1+\Delta_k^2=\Delta(P,b)-1/4+k^2$.

- It is easy to see that:

$$H\left(v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}\right) = \frac{b+b^{-1}}{4}\left(v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}\right) + \frac{-2\epsilon q^{1/2}}{b+b^{-1}}\left(v_{3/\sqrt{2}}\otimes \mathrm{W}_{\mathrm{P},b}\right)$$

But the vectors $v_{3/\sqrt{2}}$ and $v_{1/\sqrt{2}}$ are orthogonal, hence we have:

$$\widehat{\mathbb{F}}_1(P,b;q) = \left\langle v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}, H\left(v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}\right)\right\rangle = \frac{b+b^{-1}}{4}\widehat{\mathbb{F}}_0(P,b;q) \quad (8)$$

$$\widehat{\mathbb{F}}_m(P,b;q)=\left\langle v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}, H^m\left(v_{\sqrt{1/2}}\otimes \mathrm{W}_{\mathrm{P},b}\right)\right\rangle=\frac{b+b^{-1}}{4}\widehat{\mathbb{F}}_0(P,b;q)$$

Similarly applying H one can prove that:

$$\widehat{\mathbb{F}}_1(P,b;q)=\frac{b+b^{-1}}{4}\widehat{\mathbb{F}}_0(P,b;q) \tag{9}$$

$$\widehat{\mathbb{F}}_2(P,b;q)=\left(\frac{b+b^{-1}}{4}\right)^2\widehat{\mathbb{F}}_0(P,b;q), \tag{10}$$

$$\widehat{\mathbb{F}}_3(P,b;q)=\left(\frac{b+b^{-1}}{4}\right)^3\widehat{\mathbb{F}}_0(P,b;q) \tag{11}$$

$$\widehat{\mathbb{F}}_4(P,b;q)=\left(\frac{b+b^{-1}}{4}\right)^4\widehat{\mathbb{F}}_0(P,b;q)-2q\widehat{\mathbb{F}}_0(P,b;q)$$

$$\widehat{\mathbb{F}}_5(P,b;q)=\left(\frac{b+b^{-1}}{4}\right)^5\widehat{\mathbb{F}}_0(P,b;q)-\frac{17}{2}(b+b^{-1})q\widehat{\mathbb{F}}_0(P,b;q)$$

$$\widehat{\mathbb{F}}_6(P,b;q)=\left(\frac{b+b^{-1}}{4}\right)^6\widehat{\mathbb{F}}_0(P,b;q)-\frac{183(b+b^{-1})^2}{8}q\widehat{\mathbb{F}}_0(P,b;q)+$$
$$8q^{3-\Delta(P,b)}\partial_q\left(q^{\Delta(P,b)}\widehat{\mathbb{F}}'_0(P,b;q)\right)$$