The multi-dimensional Hamiltonian Structures in the Whitham method

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We consider the evolutionary systems

$$\varphi^i_t \;=\; F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) \;\equiv\; F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{x^1}, \dots, \boldsymbol{\varphi}_{x^d}, \dots) \qquad (1)$$

$i = 1, \dots, n$, $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^n)$, with d spatial dimensions, and their m-phase solutions which are usually written in the form

$$\varphi^i(\mathbf{x}, t) \;=\; \Phi^i\left(\mathbf{k}_1(\mathbf{U})\, x^1 \,+\, \dots \,+\, \mathbf{k}_d(\mathbf{U})\, x^d \,+\, \boldsymbol{\omega}(\mathbf{U})\, t \,+\, \boldsymbol{\theta}_0,\, \mathbf{U}\right) \qquad (2)$$

with some 2π-periodic in each θ^α functions

$$\Phi^i\,(\boldsymbol{\theta}, \mathbf{U}) \;\equiv\; \Phi^i\left(\theta^1, \dots, \theta^m,\, \mathbf{U}\right)$$

The functions $\mathbf{k}_q(\mathbf{U}) = (k^1_q(\mathbf{U}), \dots, k^m_q(\mathbf{U}))$ play here the role of the "wave numbers" and $\boldsymbol{\omega}(\mathbf{U}) = (\omega^1(\mathbf{U}), \dots, \omega^m(\mathbf{U}))$ represent the "frequencies" of the m-phase solutions. The parameters $\boldsymbol{\theta}_0$ represent the "initial phase shifts", which can take arbitrary values on the family of the m-phase solutions.

Here everywhere we will define a quasiperiodic function $f(\mathbf{x})$ with the fixed wave numbers $(\mathbf{k}_1, \dots, \mathbf{k}_d)$ as a function $f(\mathbf{x})$ on $\mathbb{R}^d$ coming from a smooth periodic function $f(\boldsymbol{\theta})$ on the torus $\mathbb{T}^m$:

$$f(\mathbf{k}_1 x^1 + \dots + \mathbf{k}_d x^d + \boldsymbol{\theta}_0) \quad \rightarrow \quad f(x^1, \dots, x^d)$$

under the corresponding mapping $\mathbb{R}^d \rightarrow \mathbb{T}^m$.

Let us call a smooth family of m-phase solutions of (1) any family (2) with a smooth dependence of the functions $\boldsymbol{\Phi}(\boldsymbol{\theta}, \mathbf{U})$ on some finite number of parameters $\mathbf{U} = (U^1, \dots, U^N)$.

The functions $\Phi^i(\boldsymbol{\theta}, \mathbf{U})$ are then defined by the system

$$\omega^\alpha \Phi^i_{\theta^\alpha} \, - \, F^i \left(\boldsymbol{\Phi}, \, k_1^{\beta_1} \boldsymbol{\Phi}_{\theta^{\beta_1}}, \dots, k_d^{\beta_d} \boldsymbol{\Phi}_{\theta^{\beta_d}}, \, \dots \right) \, = \, 0 \qquad (3)$$

(summation over repeated indexes).

As it is well known, in the Whitham approach the parameters $(U^1, \dots, U^N)$ become "slow" functions of coordinates and time. More precisely, we have to make the coordinate change $x^q \to X^q = \epsilon x^q$, $t \to T = \epsilon t$, $\epsilon \to 0$ and introduce the slow functions $S^\alpha(\mathbf{X}, T)$, $\alpha = 1, \dots, m$. We try to construct then the asymptotic solutions of the system

$$\epsilon \, \varphi^i_T \; = \; F^i \left(\boldsymbol{\varphi}, \, \epsilon \boldsymbol{\varphi}_{\mathbf{X}}, \, \epsilon^2 \boldsymbol{\varphi}_{\mathbf{XX}}, \, \dots \right) \tag{4}$$

with the main term having the form

$$\varphi^i_{(0)} \; = \; \Phi^i \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} + \boldsymbol{\theta}_0(\mathbf{X}, T) + \boldsymbol{\theta}, \; \mathbf{U}(\mathbf{X}, T) \right) \; , \quad i = 1, \dots, n_1 \tag{5}$$

Substituting the functions from Λ it is easy to get the relations

$$S^{\alpha}_{T} \; = \; \omega^{\alpha}(\mathbf{U}) \;\; , \quad S^{\alpha}_{X^q} \; = \; k^{\alpha}_{q}(\mathbf{U})$$

in the zero approximation, which gives the compatibility conditions

$$k^{\alpha}_{q\,T} \; = \; \omega^{\alpha}_{X^q} \;\; , \quad k^{\alpha}_{q\,X^p} \; = \; k^{\alpha}_{p\,X^q} \qquad (6)$$

for the parameters $(\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega})$ on the family Λ.
The second part of restrictions on the parameters $(U^1, \dots, U^N)$ in the Whitham method is given by the requirement of the existence of the first correction $\boldsymbol{\varphi}_{(1)}$ to solution (5)

$$\varphi^i \; \simeq \; \varphi^i_{(0)} \; + \; \epsilon \, \varphi^i_{(1)} \left(\frac{\mathbf{S}(\mathbf{X}, T)}{\epsilon} \; + \; \boldsymbol{\theta}_0(\mathbf{X}, T) \; + \; \boldsymbol{\theta}, \; \mathbf{X}, \; T \right)$$

on the space of 2π-periodic in $\boldsymbol{\theta}$ functions.

The functions $\varphi_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T)$ are defined by the linear system

$$\hat{L}_{[\mathbf{U}(\mathbf{X},T),\, \boldsymbol{\theta}_0(\mathbf{X},T)]}\, \varphi_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) \;\; = \;\; \mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$$

where $\hat{L}_{[\mathbf{U}(\mathbf{X},T),\, \boldsymbol{\theta}_0(\mathbf{X},T)]}$ is the linear operator given by the linearization of the left-hand part of system (3) on the corresponding functions from Λ and $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ is the first ϵ-discrepancy defined after the substitution of (5) in (4).

The operator $\hat{L}_{[\mathbf{U}(\mathbf{X},T),\, \boldsymbol{\theta}_0(\mathbf{X},T)]}$ represents a differential in $\boldsymbol{\theta}$ operator with periodic coefficients at every fixed $\mathbf{X}$ and T. We get then that the second part of the Whitham system should be given by the orthogonality of the function $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ to all the left eigen-vectors of $\hat{L}$ (the eigen-vectors of the adjoint operator) corresponding to the zero eigen-values at every fixed $(\mathbf{X}, T)$.

We should say, however, that the orthogonality of $\mathbf{f}_1(\theta, \mathbf{X}, T)$ to all the left eigen-vectors of $\hat{L}$ with zero eigen-values is imposed usually just in the one-phase situation. In this case we have usually just a finite number of such eigen-vectors depending regularly on the parameters $(U^1, \dots, U^N)$. The corresponding orthogonality conditions together with conditions (6) give then a regular system of hydrodynamic type which represents the Whitham system in the one-phase situation. Another important thing taking place in the one-phase situation is the possibility of constructing of all the corrections $\varphi_{(n)}$ in all orders of ϵ and representing the asymptotic solution as a regular series in integer powers of ϵ.

This situation, however, does not usually takes place in the multi-phase case where the behavior of the eigen-vectors of $\hat{L}$ is usually much more complicated. Thus, the kernels of the operators $\hat{L}$ and $\hat{L}^{\dagger}$ depend usually in highly nontrivial way on the parameters $\mathbf{U}$, being finite- or infinite-dimensional for different values of $(U^1, \dots, U^N)$. In this situation it is natural to define the "regular" orthogonality conditions just by the requirement of orthogonality of $\mathbf{f}_1$ to the "regular" set of the kernel vectors of $\hat{L}^{\dagger}$ which is usually finite also in the multi-phase case. Thus, we assume here that the kernels of the operators $\hat{L}$ and $\hat{L}^{\dagger}$ contain just a finite number of linearly independent "regular" eigen-vectors, i.e. the eigen-vectors smoothly depending on the parameters $\mathbf{U}$. The "regular" Whitham system is defined in this situation by conditions (6) and the orthogonality of the discrepancy $\mathbf{f}_1(\boldsymbol{\theta}, \mathbf{X}, T)$ to all the regular left eigen-vectors of $\hat{L}$ corresponding to the zero eigen-value.

Let us say that the first correction $\varphi_{(1)}$ to the asymptotic solution (5) can not be found here in such a simple form as in the one-phase situation. However, as the investigations of this situation show, the corrections to the main approximation $\varphi_{(0)}$ still vanish as $\epsilon \to 0$ even in the multi-phase case. So, despite the high non-triviality of the next approximation in this case, the regular Whitham system still plays very important role in consideration of slow-modulated m-phase solutions.

It is not difficult to see that the Whitham system imposes restrictions just on the functions $\mathbf{U}(\mathbf{X}, T)$ and does not contain the parameters $\boldsymbol{\theta}_0(\mathbf{X}, T)$. Indeed, the functions $\boldsymbol{\theta}_0(\mathbf{X}, T)$ can be considered just as ϵ-corrections to the functions $\mathbf{S}(\mathbf{X}, T)$, so the constraints arising on the first step include just the main terms $\mathbf{S}(\mathbf{X}, T)$, while the restrictions on $\boldsymbol{\theta}_0(\mathbf{X}, T)$ arise in the higher approximations (if they exist).

For the correct construction of the modulated solutions and a good definition of the Whitham system we have to require in fact one more thing from the family Λ. Namely, the correct procedure of constructing of modulated solutions can be implemented on the "complete regular families" Λ of m-phase solutions of (1). Let us give here the corresponding definition. Let us consider the set of parameters $\mathbf{U}$ in the form

$$\mathbf{U} = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, n^1, \dots, n^s)$$

It is easy to see then that the vectors

$$\boldsymbol{\xi}_{(\alpha)[\mathbf{U},\boldsymbol{\theta}_0]} \; = \; \boldsymbol{\Phi}_{\theta^\alpha}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \, \mathbf{U}) \;\; , \qquad \alpha = 1, \dots, m \, ,$$

$$\boldsymbol{\eta}_{(l)[\mathbf{U},\boldsymbol{\theta}_0]} \; = \; \boldsymbol{\Phi}_{n^l}(\boldsymbol{\theta} + \boldsymbol{\theta}_0, \, \mathbf{U}) \;\; , \qquad l = 1, \dots, s \, ,$$

represent regular (right) eigen-vectors of the operators $\hat{L}_{[\mathbf{U},\,\boldsymbol{\theta}_0]}$ corresponding to the zero eigen-value.

Definition 1.1.

We call a family Λ a complete regular family of m-phase solutions of (1):
1) The values $\mathbf{k}_p = (k_p^1, \dots, k_p^m)$, $\boldsymbol{\omega} = (\omega^1, \dots, \omega^m)$, *represent independent parameters on the family Λ, such that the total set of parameters of the m-phase solutions can be represented in the form*

$$(\mathbf{U}, \boldsymbol{\theta}_0) \; = \; \left(\mathbf{k}_1, \dots, \mathbf{k}_d, \, \boldsymbol{\omega}, \, n^1, \dots, n^s, \, \boldsymbol{\theta}_0\right)$$

2) The vectors $\boldsymbol{\xi}_{(\alpha)[\mathbf{U}, \boldsymbol{\theta}_0]}$ *and* $\boldsymbol{\eta}_{(l)[\mathbf{U}, \boldsymbol{\theta}_0]}$ *are linearly independent and give the maximal linearly independent set among the kernel vectors of* $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ *smoothly depending on all the parameters* **U** *on the whole set of parameters;*
3) The operator $\hat{L}_{[\mathbf{U}, \boldsymbol{\theta}_0]}$ *has exactly* $m + s$ *linearly independent left eigen-vectors with the zero eigen-value*

$$\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \;\; = \;\; \boldsymbol{\kappa}^{(q)}_{[\mathbf{k}_1, \dots, \mathbf{k}_d, \, \boldsymbol{\omega}, \, \mathbf{n}]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0) \;\; , \quad q = 1, \dots, m + s$$

among the vectors smoothly depending on the parameters **U** *on the whole set of parameters.*

By definition we will call the regular Whitham system for a complete regular family of m-phase solutions of (1) the conditions of orthogonality of the discrepancy $\mathbf{f}_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T)$ to the functions $\boldsymbol{\kappa}^{(q)}_{[\mathbf{U}(\mathbf{X},T)]}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T))$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \kappa^{(q)}_{[\mathbf{U}(\mathbf{X},T)]\, i}(\boldsymbol{\theta} + \boldsymbol{\theta}_0(\mathbf{X}, T)) \; f^i_{(1)}(\boldsymbol{\theta}, \mathbf{X}, T) \; \frac{d^m \theta}{(2\pi)^m} \;\; = \;\; 0 \qquad (7)$$

$(q = 1, \dots, m + s)$ with the compatibility conditions

$$k^\alpha_{p\,T} \;\; = \;\; \omega^\alpha_{X^p} \qquad (8)$$

$$k^\alpha_{p\,X^l} \;\; = \;\; k^\alpha_{l\,X^p} \qquad (9)$$

$\alpha = 1, \dots, m, \;\; p, l, k = 1, \dots, d,$

For our further purposes it will be convenient to separate the evolutionary part of the Whitham system and purely spatial constraints. So, let us call here relations (7) - (8) the evolutionary part of a regular Whitham system for a complete regular family Λ, while relations (9) will be considered as additional constraints for the evolutionary system (7) - (8). It is easy to see that the constraints (9) are conserved by the evolutionary system (7) - (8) being imposed at the initial time.

The evolutionary part of a regular Whitham system provides exactly $m(d+1)+s$ independent relations for $N = m(d+1)+s$ parameters $\mathbf{U} = (\mathbf{k}_1, \dots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$ at every $\mathbf{X}$ and T. In generic case the derivatives $\mathbf{U}_T$ can be expressed in terms of $\mathbf{U}_{X^l}$ from system (7) - (8) and the evolutionary part of a regular Whitham system can be written in the form

$$U^{\nu}_T \;\; = \;\; V^{\nu l}_{\mu}(\mathbf{U}) \;\; U^{\mu}_{X^l} \qquad (10)$$

$(\nu, \mu = 1, \dots, N, \;\; l = 1, \dots, d)$.

The Hamiltonian theory of systems (10) was started by B.A. Dubrovin and S.P. Novikov who introduced the concept of the Poisson bracket of Hydrodynamic Type. The local Poisson brackets of Hydrodynamic Type (Dubrovin - Novikov brackets) can be represented by the following general form

$$\{U^{\nu}(\mathbf{X}), U^{\mu}(\mathbf{Y})\} \; = \; g^{\nu\mu\, l}\,(\mathbf{U}(\mathbf{X}))\; \delta_{X^l}(\mathbf{X}{-}\mathbf{Y}) + b^{\nu\mu\, l}_{\lambda}\,(\mathbf{U}(\mathbf{X}))\; U^{\lambda}_{X^l}\,\delta(\mathbf{X}{-}\mathbf{Y}) \tag{11}$$

(summation over repeated indexes).

The theory of brackets (11) is best developed in the case of one spatial ($d = 1$) dimension. Thus, expression (11) with non-degenerate tensor $g^{\nu\mu}$ defines a Poisson bracket for $d = 1$ if and only if the tensor $g^{\nu\mu}(\mathbf{U})$ represents a flat pseudo-Riemannian (contravariant) metric on the space of parameters $\mathbf{U}$, while the functions $\Gamma^{\nu}_{\mu\gamma}(\mathbf{U}) = -\, g_{\mu\lambda}(\mathbf{U})\, b^{\lambda\nu}_{\gamma}(\mathbf{U})$ ($g^{\nu\lambda}(\mathbf{U})\, g_{\lambda\mu}(\mathbf{U}) \equiv \delta^{\nu}_{\mu}$) represent the corresponding Christoffel symbols. As a corollary, every Dubrovin - Novikov bracket in one-dimensional case can be written in the canonical (constant) form

$$\{c^{\nu}(X), c^{\mu}(Y)\} \;=\; e^{\nu}\, \delta^{\nu\mu}\, \delta'(X - Y) \;, \quad e^{\nu} = \pm 1$$

after the transition to the flat coordinates $c^{\nu} = c^{\nu}(\mathbf{U})$ for the metric $g_{\nu\mu}(\mathbf{U})$.

The functions

$$C^{\nu} \; = \; \int_{-\infty}^{+\infty} c^{\nu}(X) \; dX \quad , \qquad P \; = \; \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^{N} e^{\nu} \, (c^{\nu})^{2}(X) \; dX$$

represent the annihilators and the momentum functional of the bracket (11) for $d = 1$ respectively. The systems of Hydrodynamic Type are generated by the functionals of Hydrodynamic Type

$$H \; = \; \int_{-\infty}^{+\infty} h(\mathbf{U}) \; dX$$

according to the Dubrovin - Novikov bracket.
Let us say, that the theory of the Dubrovin - Novikov brackets in the multi-dimensional case is more complicated than in the case $d = 1$.

The Hamiltonian approach plays very important role in the theory of integrability of the Hydrodynamic Type systems in the case of one spatial dimension. Thus, according to conjecture of S.P. Novikov, all the systems of Hydrodynamic Type which can be written in the diagonal form

$$U^{\nu}_{T} \;=\; V^{\nu}(\mathbf{U})\; U^{\nu}_{X}$$

and are Hamiltonian with respect to some local bracket of Hydrodynamic Type are integrable. The Novikov conjecture was proved by S.P. Tsarev who suggested a method of integration of these systems. In fact, the method of Tsarev is applicable to a wider class of diagonalizable systems of hydrodynamic type which was called by Tsarev "semi-Hamiltonian". As it turned out later, the class of "semi-Hamiltonian systems" contains also the systems Hamiltonian with respect to generalizations of the Dubrovin - Novikov bracket - the weakly nonlocal Mokhov - Ferapontov bracket and the Ferapontov brackets.

The Hamiltonian formulation of the Whitham method was also suggested by B.A. Dubrovin and S.P. Novikov who introduced the procedure of the "averaging" of Hamiltonian structures in the theory of slow modulations. This approach is connected with the Whitham method for the evolutionary systems

$$\varphi^i_t \; = \; F^i(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots)$$

having a local field-theoretic Poisson structure

$$\{\varphi^i(x), \varphi^j(y)\} \; = \; \sum_{k \geq 0} B^{ij}_{(k)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \, \delta^{(k)}(x-y)$$

with the local Hamiltonian of the form

$$H \; = \; \int P_H(\boldsymbol{\varphi}, \boldsymbol{\varphi}_x, \dots) \, dx$$

The procedure of the averaging of local field-theoretic Poisson brackets was first developed in the case of one spatial dimension and gives a local Poisson structure of Hydrodynamic Type for the corresponding Whitham system. The method of B.A. Dubrovin and S.P. Novikov is connected with the conservative form of the Whitham system and is based on the existence of N (equal to the number of parameters U^{ν} of the family Λ) local integrals

$$I^{\nu} \; = \; \int P^{\nu}(\varphi, \varphi_x, \dots) \, dx$$

which commute with the Hamiltonian H and with each other

$$\{I^{\nu}, H\} = 0 \quad , \quad \{I^{\nu}, I^{\mu}\} = 0$$

We have then

$$P^{\nu}_t(\varphi, \varphi_x, \dots) \; \equiv \; Q^{\nu}_x(\varphi, \varphi_x, \dots)$$

for some functions $Q^{\nu}(\varphi, \varphi_x, \dots)$, while the calculation of the pairwise Poisson brackets of the densities P^{ν} gives

$$\{P^{\nu}(x), P^{\mu}(y)\} \; = \; \sum_{k \geq 0} A^{\nu\mu}_k(\varphi, \varphi_x, \dots) \, \delta^{(k)}(x-y)$$

The Dubrovin - Novikov bracket on the space of functions $\mathbf{U}(\mathbf{X})$, where $U^{\nu} \equiv \langle P^{\nu} \rangle$, is defined by the formula

$$\{U^{\nu}(X), \, U^{\mu}(Y)\} \; = \; \langle A_1^{\nu\mu} \rangle(\mathbf{U}) \, \delta'(X-Y) + \frac{\partial \langle Q^{\nu\mu} \rangle}{\partial U^{\gamma}} \; U^{\gamma}_X \, \delta(X-Y) \quad (12)$$

The Whitham system is written now in the form

$$\langle P^{\nu} \rangle_T \; = \; \langle Q^{\nu} \rangle_X \quad , \qquad \nu = 1, \dots, N$$

and can be proved to be Hamiltonian with respect to the Dubrovin - Novikov bracket (12) with the Hamiltonian

$$H_{av} \; = \; \int_{-\infty}^{+\infty} \langle P_H \rangle \, (\mathbf{U}(X)) \; dX$$

The procedure of averaging of multi-dimensional local field-theoretic Poisson brackets should be actually modified with respect to the one-dimensional case.

We assume now that system (1) is Hamiltonian with respect to a local field-theoretic Poisson bracket

$$\{\varphi^i(\mathbf{x}), \varphi^j(\mathbf{y})\} = \sum_{l_1,\dots,l_d} B^{ij}_{(l_1,\dots,l_d)}(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \dots) \, \delta^{(l_1)}(x^1-y^1) \, \dots \, \delta^{(l_d)}(x^d-y^d) \tag{13}$$

$(l_1, \dots, l_d \geq 0)$, with a local Hamiltonian of the form

$$H = \int P_H(\boldsymbol{\varphi}, \boldsymbol{\varphi}_{\mathbf{x}}, \boldsymbol{\varphi}_{\mathbf{xx}}, \dots) \; d^d x \tag{14}$$

Like in the Dubrovin - Novikov procedure we have to require here the existence of N (equal to the number of parameters $(\mathbf{k}_1, \ldots, \mathbf{k}_d, \boldsymbol{\omega}, \mathbf{n})$) first integrals

$$I^{\nu} \; = \; \int P^{\nu} \left(\varphi, \, \varphi_{\mathbf{x}}, \, \varphi_{\mathbf{xx}}, \ldots \right) \, d^d x \tag{15}$$

such that their values can be chosen as the parameters $(U^1, \ldots, U^N)$ on the family Λ. We assume also that all the integrals I^{ν} commute with each other and with the Hamiltonian H

$$\{ I^{\nu} \, , \, I^{\mu} \} \, = \, 0 \quad , \quad \{ I^{\nu} \, , \, H \} \, = \, 0 \tag{16}$$

according to bracket (13). For the time evolution of the densities $P^{\nu}(\mathbf{x})$ we can write

$$P^{\nu}_t \left(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \ldots \right) \; = \; Q^{\nu 1}_{x^1} \left(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \ldots \right) + \ldots + Q^{\nu d}_{x^d} \left(\varphi, \varphi_{\mathbf{x}}, \varphi_{\mathbf{xx}}, \ldots \right)$$

with some functions $Q^{\nu l}$.

In fact, we have to put also some additional requirements on the family Λ and the set of the integrals I^{ν}. Namely, we have to require that the family Λ represents a regular Hamiltonian family of m-phase solutions of system (1) and the set $(I^1, \dots, I^N)$ represents a complete Hamiltonian set of commuting integrals. So, the family Λ should in fact satisfy the following requirements:

1) The family Λ represents a complete regular family of m-phase solutions of (1) in the sense of Definition 1.1;

2) The corresponding bracket (13) has on Λ constant number of annihilators N^1, ..., N^s with linearly independent variation derivatives $\delta N^l / \delta \varphi^i(\mathbf{x})$.

In the similar way, we have to put also the following requirements on the set $(I^1, \ldots, I^N)$:

1) The restriction of the functionals $(I^1, \ldots, I^N)$ on the quasiperiodic solutions of the family Λ gives a complete set of parameters $(U^1, \ldots, U^N)$ on this family;

2) The Hamiltonian flows generated by $(I^1, \ldots, I^N)$ generate on Λ linear phase shifts of $\boldsymbol{\theta}_0$ with frequencies $\boldsymbol{\omega}^{\nu}(\mathbf{U})$, such that

$$\mathrm{rk}\ ||\omega^{\alpha\nu}(\mathbf{U})|| \;=\; m$$

3) The linear space generated by the variation derivatives $\delta I^{\nu}/\delta\varphi^{i}(\mathbf{x})$ on Λ contains the variation derivatives of all the annihilators N^q of the bracket (13), such that

$$\left.\frac{\delta N^l}{\delta\varphi^i(\mathbf{x})}\right|_{\Lambda} \;=\; \sum_{\nu=1}^{N} \gamma^l_\nu(\mathbf{U}) \;\left.\frac{\delta I^\nu}{\delta\varphi^i(\mathbf{x})}\right|_{\Lambda}$$

for some smooth functions $\gamma^l_\nu(\mathbf{U})$ on the family Λ.

Under the requirements formulated above the set $(I^1, \ldots, I^N)$ can be used for construction of a local field-theoretic Poisson bracket for the regular Whitham system on a regular Hamiltonian family Λ of m-phase solutions of system (1). The corresponding procedure in the absence of the pseudo-phases can be formulated in the following way:
Let us represent the pairwise Poisson brackets of the densities $P^\nu(\mathbf{x})$, $P^\mu(\mathbf{y})$ in the form

$$\{P^\nu(\mathbf{x}) \, , \, P^\mu(\mathbf{y})\} \; = \; \sum_{l_1,\ldots,l_d} A^{\nu\mu}_{l_1\ldots l_d}(\varphi, \varphi_{\mathbf{x}}, \ldots) \; \delta^{(l_1)}(x^1 - y^1) \, \ldots \, \delta^{(l_d)}(x^d - y^d)$$

$(l_1, \ldots, l_d \geq 0)$. According to relations (16) we can also write here the relations

$$A^{\nu\mu}_{0\ldots0}(\varphi, \varphi_{\mathbf{x}}, \ldots) \;\; \equiv \;\; \partial_{x^1} \, Q^{\nu\mu 1}(\varphi, \varphi_{\mathbf{x}}, \ldots) \, + \, \ldots \, + \, \partial_{x^d} \, Q^{\nu\mu d}(\varphi, \varphi_{\mathbf{x}}, \ldots)$$

for some functions $(Q^{\nu\mu 1}, \ldots, Q^{\nu\mu d})$.

Let us say, however, that the averaged Poisson bracket does not have in general the form (11) for $d > 1$, which is connected with the fact that the Hamiltonian structure should be defined now just on the "submanifold" in the space of functions $\mathbf{U}(\mathbf{X})$, given by the constraints $k^{\alpha}_{q X^p} = k^{\alpha}_{p X^q}$, $\alpha = 1, \dots, m$, $q, p = 1, \dots, d$. To define the corresponding Poisson bracket we have to introduce the coordinates $S^{\alpha}(\mathbf{X})$ $(\alpha = 1, \dots, m)$ on this submanifold, defined by the relations $S^{\alpha}_{X^q} = k^{\alpha}_q(\mathbf{X})$. It is easy to see, that the spatial derivatives of the functions $S^{\alpha}(\mathbf{X})$ provide just md coordinates on the family Λ, connected with the wave numbers of the solutions. For the rest $m + s$ coordinates we can use just arbitrary independent values U^{γ}, $\gamma = 1, \dots, m + s$ from the full set $U^{\nu} = \langle P^{\nu} \rangle$, $\nu = 1, \dots, N$ on Λ. The corresponding regular Whitham system on Λ can then be written in the absence of pseudo-phases in the form:

$$
\begin{aligned}
S^{\alpha}_T &= \omega^{\alpha}\left(\mathbf{S}_{\mathbf{X}}, U^1, \dots, U^{m+s}\right) \quad , \quad \alpha = 1, \dots, m \ , \\
U^{\gamma}_T &= \langle Q^{\gamma 1} \rangle_{X^1} + \dots + \langle Q^{\gamma d} \rangle_{X^d} \quad , \quad \gamma = 1, \dots, m + s \ ,
\end{aligned}
\tag{17}
$$

where $\langle Q^{\gamma p} \rangle = \langle Q^{\gamma p} \rangle(\mathbf{S}_{\mathbf{X}}, U^1, \dots, U^{m+s})$.

It can be shown then that the Hamiltonian structure of system (17) is given by the Poisson bracket

$$\Big\{ S^{\alpha}(\mathbf{X}) \, , \, S^{\beta}(\mathbf{Y}) \Big\} \; = \; 0 \; ,$$

$$\{ S^{\alpha}(\mathbf{X}) \, , \, U^{\gamma}(\mathbf{Y}) \} \; = \; \omega^{\alpha\gamma} \left(\mathbf{S_X}, U^1(\mathbf{X}), \ldots, U^{m+s}(\mathbf{X}) \right) \, \delta(\mathbf{X} - \mathbf{Y}) \; , \qquad (18)$$

$$\begin{aligned} \{ U^{\gamma}(\mathbf{X}) \, , \, U^{\rho}(\mathbf{Y}) \} \; &= \; \langle A^{\gamma\rho}_{10\ldots0} \rangle \left(\mathbf{S_X}, \, U^1(\mathbf{X}), \ldots, U^{m+s}(\mathbf{X}) \right) \; \delta_{X^1}(\mathbf{X} - \mathbf{Y}) \; + \\ &+ \; \ldots \; + \; \langle A^{\gamma\rho}_{0\ldots01} \rangle \left(\mathbf{S_X}, \, U^1(\mathbf{X}), \ldots, U^{m+s}(\mathbf{X}) \right) \; \delta_{X^d}(\mathbf{X} - \mathbf{Y}) \; + \\ &+ \; \left[\langle Q^{\gamma\rho \, p} \rangle \left(\mathbf{S_X}, \, U^1(\mathbf{X}), \ldots, U^{m+s}(\mathbf{X}) \right) \right]_{X^p} \; \delta(\mathbf{X} - \mathbf{Y}) \; , \quad \gamma, \rho = 1, \ldots, m+s \end{aligned}$$

with the Hamiltonian functional

$$H_{av} \; = \; \int \langle P_H \rangle \; \left(\mathbf{S_X}, \, U^1(\mathbf{X}), \ldots, U^{m+s}(\mathbf{X}) \right) \; d^d X$$