F-theorem and double trace deformations of three-dimensional large N CFT

Grigory Tarnopolsky

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in collaboration with: S. Giombi, I. Klebanov, S. Pufu, B. Safdi

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Renorm Group theorems

Two dimensions — 2d

A.B. Zamolodchikov, 1987 — c-theorem:

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{12} R, \qquad c_{IR} < c_{UV}$$

Four dimensions — 4d

J.L. Cardy, 1988 — Proposal for a-theorem,
Z. Komargodski, A. Schwimmer, 2011 — proof.

$$\langle T^{\mu}_{\mu} \rangle = aE_4 - cW^2_{\mu\nu\rho\sigma}, \qquad a_{IR} < a_{UV}$$

$$E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$$

Three dimensions — 3d

Jafferis, Klebanov, Pufu, Safdi, 2011 — Proposal for F-theorem,

$$F = -\log |Z_{S_3}|, \qquad F_{IR} < F_{UV}$$

Plan of presentation

• Simple check of the F-theorem for double trace deformed theory by scalar operator: Calculation of $\delta F = F_{UV} - F_{IR}$

• Calculation of $\delta F = F_{UV} - F_{IR}$ using dual description (AdS_4)

• Check of the F-theorem for double trace deformed theory by higher spin currents: Calculation of $\delta F = F_{UV} - F_{IR}$

• General dual description check

Double trace deformation for scalar field

Consider the action

$$Z = \int D\phi \exp\left(-S_0 - \frac{\lambda_0}{2}\int d^d x \sqrt{G}\Phi^2\right)$$

where Φ is relevant operator and it has dimension $\Delta \in (d/2 - 1, d/2)$ in UV theory, λ_0 is the bare coupling defined at the UV scale μ_0. S_0 is some conformal action.

We can write

$$Z = Z_0 \left\langle \exp\left(-\frac{\lambda_0}{2}\int d^d x \sqrt{G}\Phi^2\right)\right\rangle_0$$

One can get

$$\delta F = F_{IR} - F_{UV} = -\log\left|\frac{Z}{Z_0}\right|$$

Hubbard-Stratonovich trasformation

Introduce auxiliary field σ

$$\frac{Z}{Z_0} = \int D\sigma \left\langle \exp\left[\int d^d x \sqrt{G}\left(\frac{1}{2\lambda_0}\sigma^2 + \sigma\Phi\right)\right]\right\rangle_0$$

Using of large N

Large N implies that the higher point functions of Φ are suppressed by relative to the two-point function by factors of $1/N$:

$$\left\langle \exp\left[\int d^d x \sqrt{G}\left(\frac{1}{2\lambda_0}\sigma^2 + \sigma\Phi\right)\right]\right\rangle_0 = \exp\left[\frac{1}{2}\left\langle\left(\int d^d x \sqrt{G}\sigma\Phi\right)^2\right\rangle_0\right]$$

Now we have dynamics for σ, and one can get for δF_Δ

$$\delta F_\Delta = \frac{1}{2}\mathrm{tr}\log(K),$$

where

$$K(x,y) = \frac{1}{\sqrt{G(x)}}\delta(x-y) + \lambda_0\langle\Phi(x)\Phi(y)\rangle_0.$$

Futher calculations

We have

$$\langle \Phi(x)\Phi(y)\rangle_0 = \frac{1}{\sqrt{2}}\frac{1}{s(x,y)^{2\Delta}},$$

where $s(x,y)$ is chordal distance $s(x,y) = |\vec{x} - \vec{y}|$, $\vec{x}, \vec{y} \in S^d$.

Expand $s(x,y)^{-2\Delta}$ in spherical harmonics

$$\frac{1}{s(x,y)^{2\Delta}} = \frac{1}{R^{2\Delta}} \sum_{n,m} k_n Y^*_{nm}(x) Y_{nm}(y)$$

where

$$k_n = \pi^{d/2} 2^{d-\Delta} \frac{\Gamma(\frac{d}{2} - \Delta)}{\Gamma(\Delta)} \frac{\Gamma(n+\Delta)}{\Gamma(d+n-\Delta)}, \quad n \geqslant 0.$$

One can get now

$$\delta F_\Delta = \frac{1}{2} \sum_{n=0}^{\infty} m_n \log[1 + \lambda_0 R^{d-2\Delta} k_n]$$

IR limit

Because $d - 2\Delta > 0$ in the IR limit $R^{d-2\Delta} \to \infty$.

$$\delta F_\Delta = \frac{1}{2} \sum_{n=0}^{\infty} m_n \log \left(\frac{\Gamma(n+\Delta)}{\Gamma(d+n-\Delta)} \right),$$

where $m_n = \frac{(2n+d-1)(n+d-2)!}{(d-1)!n!}$.

To calculate this sum, consider $\frac{d(\delta F)}{d\Delta}$:

For $d = 3$ we find

$$\frac{d(\delta F)}{d\Delta} = -\frac{\pi}{6}(\Delta - 1)(\Delta - \frac{3}{2})(\Delta - 2)\cot(\pi\Delta)$$

Thus

$$\delta F_\Delta = -\frac{\pi}{6} \int_\Delta^{3/2} dx (x-1)(x-\frac{3}{2})(x-2)\cot(\pi x)$$

Particular Examples

$\Delta = 1$: $O(N)$ model

$$\delta F_{\Delta=1} = -\frac{\zeta(3)}{8\pi^2} \approx -0.0152,$$

where the action is

$$S_{O(N)} = \int d^3x \left[\frac{1}{2}(\partial_\mu \phi^a)^2 + \frac{\lambda_0}{2N}(\phi^a\phi^a)^2\right]$$

$\Delta = 1/2$: mass term to free scalar field

$$\delta F_{\Delta=1/2} = -\frac{1}{16}\left(2\log 2 - \frac{3\zeta(3)}{\pi^2}\right) \approx -0.0638$$

Calculation δF_Δ using dual description in AdS_4

I. Klebanov, A. Polyakov, 2002

$$O(N) \text{ vector models} \leftrightarrow \text{Vasiliev higher-spin gauge theory in } \mathrm{AdS}_4$$

Euclidean version of AdS_d can be described by the metric

$$ds^2 = \frac{4\sum_{i=0}^{d} dy_i^2}{(1-|y|^2)^2} = \frac{dz^2 + \sum_{i=1}^{d} dx_i^2}{z^2},$$

where $\sum_{i=0}^{d} y_i^2 < 1$. Its boundary is the S^d sphere: $\sum_{i=0}^{d} y_i^2 = 1$.

General references on AdS/CFT are

- J. Maldacena, 1998
- S. Gubser, I. Klebanov, A. Polyakov, 1988
- E. Witten, 1988.

The calculation in AdS. I

Consider a free massive scalar field in AdS_{d+1}

$$(\nabla^2 + 2(d-2) - m^2)h(\vec{x}, z) = 0$$

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A solution of this equation near a boundary $z = 0$, behaves as

$$h(\vec{x}, z) \sim z^{\Delta} f(\vec{x}),$$

where Δ is a root of equation

$$(\Delta - 2)(\Delta + 2 - d) = m^2.$$

The Solutions are

$$\Delta_{\pm} = \frac{d}{2} \pm \nu, \quad \nu = \sqrt{m^2 + (\frac{d}{2} - 2)^2}.$$

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The calculation in AdS. II

The same bulk theory describes two different CFTs, depending on the boundary conditions for the field $h(z, \vec{x})$.

- boundary condition $h \sim z^{\Delta_-}$ corresponds to UV CFT
- boundary condition $h \sim z^{\Delta_+}$ corresponds to IR fixed point.

For the free energy F we have

$$F_{\Delta_\pm} = -\log \int Dh e^{-S[h]}\Big|_{\Delta_\pm}.$$

Thus for δF_Δ we have

$$\delta F_\Delta = F_{\Delta_-} - F_{\Delta_+} = \frac{1}{2}\left[\mathrm{tr}\log_-(-\nabla^2 + m^2) - \mathrm{tr}\log_+(-\nabla^2 + m^2)\right]$$

The calculation in AdS. III

Take a derivative with respect to Δ

$$\frac{d(\delta F_\Delta)}{d\Delta} = (2\Delta - d)\frac{\partial \delta F_\Delta}{\partial m^2} = \frac{2\Delta - d}{2}\int \mathrm{vol}_{AdS_{d+1}}(G_{\Delta_-}(x,x) - G_{\Delta_+}(x,x))$$

Propagator:

$$G_\Delta(x,x) = \frac{\Gamma(\Delta)}{2^{\Delta+1}\pi^{d/2}\Gamma(1+\Delta-\frac{d}{2})}F(\frac{\Delta}{2},\frac{1+\Delta}{2},1+\Delta-\frac{d}{2},1)$$

Regularized AdS volume:

$$\int \mathrm{vol}_{H^{d+1}} = \pi^{d/2}\Gamma(-\frac{d}{2}).$$

Finally one finds

$$\frac{d\delta F_\Delta}{d\Delta} = -\frac{\pi}{6}(\Delta-1)(\Delta-\frac{3}{2})(\Delta-2)\cot(\pi\Delta)$$

Double trace deformation for general spin s

Consider the following action action on S^3

$$S = S_0 + \frac{\lambda_0}{2}\int d^3x\sqrt{g}J_{\mu_1\ldots\mu_s}(x)J^{\mu_1\ldots\mu_s}(x),$$

where $J_{\mu_1\ldots\mu_s}$ is a symmetric traceless tensor.

- This theory has a UV fixed point where $J_{\mu_1\ldots\mu_s}$ has dimension $\Delta_- = 3-\Delta+O(1/N)$.

Result

$$\delta F_\Delta^{(s)} = -\frac{\pi(2s+1)}{6}\int_{3/2}^{\Delta}(x-3/2)(x+s-1)(x-s-2)\cot(\pi x).$$

Thank you for your attention!