Correlation functions in Minimal Liouville Gravity from Douglas string equation

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- Our starting point is **Conjecture**: Partition function of Liouville Minimal gravity satisfies Douglas string equation and KdV equations.
- Natural parameters of Liouville gravity are connected to KdV times by special non-linear relation.
- We found these relations simultaneously with correlation functions, which obey the conformal selection rules.

2D quantum gravity

- (q,p) Minimal Liouville Gravity consists of Liouville field $\phi(x)$ and CFT (q,p) Minimal Model.
- In (q,p) Minimal Liouville Gravity we are interested in evaluating the correlators of observables O_{mn}

$$\langle O_{m_1 n_1} \dots O_{m_k n_k} \rangle$$

where

$$O_{m,n} = \int \mathcal{O}_{m,n}(x) d^2 x, \qquad \mathcal{O}_{m,n}(x) = \Phi_{m,n} e^{2b\delta_{m,n}\phi(x)}$$

and $\delta_{m,n}$ is the gravitational dimension

$$\delta_{m,n} = \frac{p+q-|pm-qn|}{2q}, \qquad O_{m,n} \sim \mu^{\delta_{m,n}}$$

- Instead of the correlators it is convenient to study their generating function

$$Z^{MG}(\lambda) = \langle \exp(\sum_{m,n} \lambda_{m,n} O_{m,n}) \rangle =$$

$$= Z_0 + \sum \lambda_{m,n} \langle O_{mn} \rangle + \frac{1}{2} \sum \lambda_{m_1,n_1} \lambda_{m_2,n_2} \langle O_{m_1 n_1} O_{m_2 n_2} \rangle + \dots$$

- The generating function possesses a definite mass dimension

$$Z^{MG} \sim \mu^{1+\frac{p}{q}}$$

- In minimal conformal field theory there are "selection rules"

$$
\begin{aligned}
&\langle \Phi_{m,n} \rangle = 0, && (m,n) \neq (1,1), \\
&\langle \Phi_{m_1 n_1} \Phi_{m_2 n_2} \rangle = 0, && (m_1, n_1) \neq (m_2, n_2), \\
&\langle \Phi_{m_1 n_1} \Phi_{m_2 n_2} \Phi_{m_3 n_3} \rangle = 0, && \begin{cases} m_{i_1} > \min(m_{i_2} + m_{i_3}, 2p - m_{i_2} - m_{i_3} - 4) \\ n_{i_1} > \min(n_{i_2} + n_{i_3}, 2p - n_{i_2} - n_{i_3} - 4) \end{cases} \\
&\dots
\end{aligned}
$$

for arbitrary permutation (i_1, i_2, i_3) of numbers $(1, 2, 3)$.

Contact terms and resonance relations

- In local field theory the integrated correlation functions suffer from intrinsic ambiguity from the contact terms. The integral

$$\int_{x_1,\ldots,x_k} \langle \mathcal{O}_{m_1,n_1}(x_1)\ldots\mathcal{O}_{m_k,n_k}(x_k)\rangle$$

may pick up contributions from delta like terms in the integrand, when two or more points x_i collide.

- One may add to the n-point correlation numbers some k-point correlation numbers. For instance

$$\langle O_{m_1,n_1}O_{m_2,n_2}\rangle \to \langle O_{m_1,n_1}O_{m_2,n_2}\rangle + \sum_{m,n} A_{m,n}^{(m_1n_1)(m_2n_2)}\langle O_{m,n}\rangle.$$

Or equivalently

$$\lambda_{m,n} \to \lambda_{m,n} + \sum_{m_1,n_1,m_2,n_2} A_{m,n}^{(m_1n_1)(m_2n_2)}\lambda_{m_1n_1}\lambda_{m_2n_2}.$$

- The form of these relations can be further restricted by scaling invariance. Namely since the fields $O_{m,n}$ and times $\lambda_{m,n}$ have definite gravitational dimensions

$$O_{m,n}\sim\mu^{\delta_{m,n}}, \qquad \lambda_{m,n}\sim\mu^{-\delta_{m,n}}$$

then for instance $A_{m,n}^{(m_1n_1)(m_2n_2)}\neq 0$ only if $\delta_{m,n}=\delta_{m_1,n_1}+\delta_{m_2,n_2}$.

- **CONJECTURE**: There exists such a change of variables $\{\lambda_{m,n}\} \to \{t_{k,\alpha}\} \equiv \{t_{m,n}\}$, $Z^{MG}(\lambda) = Z(t(\lambda))$ that Z(t) satisfies the equation

$$\frac{\partial^2 Z}{\partial x \partial t_{0,\alpha}} = \text{Res} Q^{\frac{\alpha}{q}} \Big|_{u^\alpha = u^\alpha_*}$$

where u^α_* is the appropriate solution of the "String equation"

$$\{P, Q\} = \frac{\partial P}{\partial x}\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y}\frac{\partial Q}{\partial x} = 1$$

and

$$x = t_{0,1}, \qquad Q = y^q + \sum_{\alpha=1}^{q-1} u_\alpha(x) y^{q-\alpha-1}, \qquad P = \left(Q^{\frac{p}{q}} + \sum_{\alpha=1}^{q-1} \sum_{k=1} t_{k,\alpha} Q^{k+\frac{\alpha}{q}} \right)_+$$

$(\ldots)_+$ denotes that non-negative powers of y are taken in the series expansion.

- Then the function $Z(t)$ has the same scaling properties as $Z^{MG}(\lambda)$

$$Z \sim y^{2q(1+\frac{p}{q})}, \qquad Z^{MG} \sim \mu^{1+\frac{p}{q}} \Rightarrow y \sim \mu^{\frac{1}{2q}}$$

- The times $t_{k,\alpha}$ are related to the times $t_{m,n}$. One must redefine the sum in H

$$\sum_{k,\alpha} t_{k,\alpha} ResQ^{k+\frac{\alpha}{q}} = \sum_{m,n} t_{m,n} Q^{\frac{|pm-qn|}{q}}$$

- We establish the explicit form of the relation between $\lambda_{m,n}$ and $t_{m,n}$ (the resonance relations) by demanding that the correlation functions satisfy the fusion rules.

- From the definition of $Z(t)$ follows that it is the logarithm of the Sato tau-function. It is convenient to make use of the following representation for it

$$Z = \int_0^{u_*} C_\alpha^{\beta\gamma} \frac{\partial H}{\partial u_\beta} \frac{\partial H}{\partial u_\gamma} du^\alpha$$

where u_*^α is an appropriate solution of the string equation and

$$H = \text{Res} Q^{\frac{p}{q}+1} + \sum_{m,n} t_{m,n} \text{Res} Q^{\frac{|pm-qn|}{q}}$$

$$C_{\alpha\beta\gamma} = \text{Res} \frac{\Phi_\alpha \Phi_\beta \Phi_\gamma}{Q'}, \qquad g_{\alpha\beta} = \text{Res} \frac{\Phi_\alpha \Phi_\beta}{Q'},$$

$$C_\alpha^{\beta\gamma} = C_{\alpha\rho\delta} g^{\rho\beta} g^{\delta\gamma}$$

where $\Phi_\alpha = y^{q-\alpha-1}$, $Q' = \frac{dQ}{dy}$.

- This construction is deeply related to some associative commutative algebra - WDVV algebra

- WDVV algebra is the algebra of polynomials $P[y]$ mod Q'.
- If we take the basis in DVV algebra $\Phi_\alpha = \frac{\partial Q}{\partial u^\alpha}$, then

$$\Phi_\alpha \Phi_\beta = C_{\alpha\beta}^{\gamma} \Phi_\gamma \mod Q'$$

- There is a natural scalar product on this algebra

$$g_{\alpha\beta} = \langle \Phi_\alpha \Phi_\beta \rangle = \text{res} \frac{\Phi_\alpha \Phi_\beta}{Q'}$$

- The constants, appearing in the integral, are related to the structure constants

$$C_\alpha^{\beta\gamma} = C_{\alpha\rho}^{\beta} g^{\rho\gamma}$$

- The integral does not depend on the path of integration, since the form

$$\Omega = C_\alpha^{\beta\gamma} \frac{\partial H}{\partial u_\beta} \frac{\partial H}{\partial u_\gamma} du^\alpha$$

is closed. This is true because of two properties:
1). Associativity: $C_{\alpha\beta}^{\gamma} C_{\gamma\delta}^{\phi} = C_{\alpha\gamma}^{\phi} C_{\beta\delta}^{\gamma}$
2). Recursion relation: $\frac{\partial^2 H_{n,\alpha}}{\partial v_\beta \partial v_\gamma} = C_{\beta\gamma}^{\delta} \frac{\partial H_{n-1,\alpha}}{\partial v_\delta}$, where $H_{n,\alpha} = resQ^{n+\frac{\alpha}{q}}$

Explicit examples: $(p, q) = (2s+1, 2)$

- In this case one has $Q = y^2 + u$ and

$$P(u) := \frac{\partial H}{\partial u} = u^{s+1} + t_0 u^{s-1} + \sum_{k=1}^{s-1} t_k u^{s-k-1}$$

$$Z = \frac{1}{2}\int_0^{u_*} P^2(u)du$$

where u_* is the appropriate zero of the polynomial $P(u)$.

- To get the generating function for the correlation numbers one inserts the resonance relations

$$t_k = \lambda_k + A_k \mu^{\delta_k} + B_k^{k_1}\lambda_{k_1}\mu^{\delta_k - \delta_{k_1}} + C_k^{k_1 k_2}\lambda_{k_1}\lambda_{k_2}\mu^{\delta_k - \delta_{k_1} - \delta_{k_2}} + \dots$$

into the partition function and polynomial $P(u)$. The result is of the form

$$Z = Z_0 + \sum_{k=1}^{s-1}\lambda_k Z_k + \frac{1}{2}\sum_{k_1,k_2=1}^{s-1}\lambda_{k_1}\lambda_{k_2}Z_{k_1 k_2} + \dots$$

$$P = P_0 + \sum_{k=1}^{s-1}\lambda_k P_k + \frac{1}{2}\sum_{k_1,k_2=1}^{s-1}\lambda_{k_1}\lambda_{k_2}P_{k_1 k_2} + \dots$$

- Also from the original form of the polynomial $P(u)$ one finds that

$$P_0(u) = u^{s+1} + A\mu u^{s-1} + B\mu^2 u^{s-3} + \dots$$
$$P_k(u) = u^{s-k-1} + C\mu u^{s-k-3} + D\mu^2 u^{s-k-5} + \dots$$
$$P_{k_1 k_2} = u^{s-k_1-k_2-3} + E\mu u^{s-k_1-k_2-5} + F\mu^2 u^{s-k_1-k_2-k_3-7} + \dots$$

- The dimensions are

$$\lambda_k \sim \mu^{\frac{k+2}{2}}, \qquad Z \sim \mu^{\frac{2s+3}{2}}, \qquad Z_{k_1 \dots k_n} \sim \mu^{\frac{2s+3-\sum(k_i+2)}{2}}$$

- As usually in the spirit of the scaling theory, we are interested only in the singular part of partition function and disregard the regular part as non-universal.

- After an appropriate renormaliztion, which we do not present, we switch to dimensionless quantities.
- One finds one- and two-point correlation numbers

$$\mathcal{Z}_k = \int_0^1 du P_0(u) P_k(u)$$
$$\mathcal{Z}_{k_1 k_2} = \int_0^1 du (P_{k_1}(u) P_{k_2}(u) + P_0(u) P_{k_1 k_2}(u))$$

- The second term in the two point numbers is actually absent since the polynomial $P_{k_1 k_2}$ could be written as linear combination of P_k.
- Also it is convenient here to introduce a new variable y instead of u

$$\frac{y+1}{2} = u^2, \qquad du = \frac{dy}{2\sqrt{2}(1+y)^{\frac{1}{2}}}$$

In terms of the variable y the polynomials $P_0, P_k, \dots$ will contain all powers instead of going with step 2. And the shift was made in order for the interval of integration be $[-1, 1]$ instead of $[0, 1]$.

- Fusion rules determine the polynomials Q:

s-even	$P_0(y) = P^{(0,-\frac{1}{2})}_{\frac{s+1}{2}}(y) - P^{(0,-\frac{1}{2})}_{\frac{s-1}{2}}(y)$
s-odd	$P_0(y) = u\left(P^{(0,\frac{1}{2})}_{\frac{s}{2}}(y) - P^{(0,\frac{1}{2})}_{\frac{s-2}{2}}(y)\right)$
(s+k)-even	$P_k(y) = P^{(0,-\frac{1}{2})}_{\frac{s-k-1}{2}}(y)$
(s+k)-odd	$P_k(y) = uP^{(0,\frac{1}{2})}_{\frac{s-k-2}{2}}(y)$

where $P_n^{(a,b)}$ is Jacobi polynomial.

- Due to the relation between Jacobi polynomials ans Legendre polynomials P_n

$$P_n^{(0,-\frac{1}{2})}(2x^2-1) = P_{2n}(x)$$
$$xP_n^{(0,\frac{1}{2})}(2x^2-1) = P_{2n+1}(x)$$

it is in agreement with the paper A.Belavin, A. Zamolodchikov (2008).

- In this case the polynomial Q and the action are

$$Q = y^3 + uy + v, \qquad H_{k.\alpha} := \text{res} Q^{k+\frac{\alpha}{q}}$$
$$H(u,v) = H_{s+1,\alpha} + \sum_{k=0}^{s-1} t_k H_{s-k-1,\alpha} + \sum_{k=s}^{3s+\alpha-2} t_k H_{k-s,3-\alpha}.$$

- One takes then an appropriate solution (u_*, v_*) of the equations

$$\begin{cases} H_u = 0 \\ H_v = 0 \end{cases}$$

where the lowered indices u, v denote the derivatives over u and v.

- The free energy for this model is defined as

$$Z = \frac{1}{2}\int (H_u^2 - \frac{u}{3} H_v^2) du + \int H_u H_v dv$$

where the integration contour goes from $(u,v) = (0,0)$ to $(u,v) = (u_*, v_*)$.

- A new feature in this case is that for $(3s+\alpha,3)$ MG polynomials depend on two variables u, v instead of one variable u in the case $(2s+1,2)$ MG.
- However it turns out to be possible to find the solution of the string equations such that

$$u_*(\lambda=0)=u_0, \qquad v_*(\lambda=0)=0$$

and it ensures that one point correlation functions turn to zero.
- This allows to reduce the problem to determination of polynomials depending on one variable.
- Namely we will show that one the functions either $H_u(\lambda=0)$ or $H_v(\lambda=0)$ is always odd in v and thus turns to zero at $v=0$.

- Substitute the resonance relations into the action. We get some expression which could be written as expansion over λ_k

$$H(u,v) = H^0(u,v) + \sum_{k=1}^{3s+\alpha-2} \lambda_k H^k(u,v) + \frac{1}{2}\sum_{k_1,k_2=1}^{3s+\alpha-2} \lambda_{k_1}\lambda_{k_2} H^{k_1k_2}(u,v) + \ldots$$

and dimensional analysis gives

$$H^0 = \sum_{n,m} u^n v^m \sim \mu^{\frac{s+1}{2}+\frac{\alpha+1}{6}}$$

And since $u \sim \mu^{\frac{1}{3}}, v \sim \mu^{\frac{1}{2}}$, the parity with respect to v is

$$H^0(u,-v) = (-1)^{s+\alpha} H^0(u,v)$$

- Thus, depending on the values of s and α, either $\frac{\partial H^0}{\partial u}$ or $\frac{\partial H^0}{\partial v}$ is odd function of v. Consequently, $v=0$ is always a solution of string equations at $\lambda = 0$.
- Similarly a bunch of functions among $H^k, H^{k_1k_2}, \ldots$ are odd in v and thus turn to zero at $v=0$. This ensures that only polynomials of one variable u need to be defined by fusion rules.

- The following arguments are similar to those in $(2s+1,2)$ case.
- For instance from the fusion rules for one- and two-point correlators

$(s+\alpha)$ - even	$H_u^0 = x^{2(\alpha-1)}\left(P^{(0,\frac{2}{3}(2\alpha-3))}_{\frac{s-\alpha+2}{2}}(y) - P^{(0,\frac{2}{3}(2\alpha-3))}_{\frac{s-\alpha}{2}}(y)\right)$
$(s+\alpha)$ - odd	$H_v^0 = x^{2\alpha-1}\left(P^{(0,\frac{1}{3}(4\alpha-3))}_{\frac{s-\alpha+1}{2}}(y) - P^{(0,\frac{1}{3}(4\alpha-3))}_{\frac{s-\alpha-1}{2}}(y)\right)$
$(s+\alpha+k)$ - even, $k<s$	$H_u^k = x^{2(\alpha-1)}P^{(0,\frac{2}{3}(2\alpha-3))}_{\frac{s-k-\alpha}{2}}(y)$
$(s+\alpha+k)$ - odd, $k<s$	$H_v^k = x^{2\alpha-1}P^{(0,\frac{1}{3}(4\alpha-3))}_{\frac{s-k-\alpha-1}{2}}(y)$
$(s+\alpha+k)$ - even, $k\geq s$	$H_u^k = x^{2(2-\alpha)}P^{(0,\frac{2}{3}(3-2\alpha))}_{\frac{k-s+\alpha-2}{2}}(y)$
$(s+\alpha+k)$ - odd, $k\geq s$	$H_v^k = x^{5-2\alpha}P^{(0,\frac{1}{3}(9-4\alpha))}_{\frac{k-s+\alpha-4}{2}}(y)$

where all the polynomials are presented at $v=0$, $P_n^{(a,b)}$ - are again Jacobi polynomials and

$$x = \frac{u}{u_0}, \qquad \frac{x+1}{2} = y^3$$

- Let us concentrate on the special case when $s+\alpha$ – even and all k – even. In this case

$$Z_{k_1k_2k_3}=-\frac{H_u^{k_1}H_u^{k_2}H_u^{k_3}}{(H_u^0)'}\Big|_{u=1}+\int_0^1 H_u^0H_u^{k_1k_2k_3}du+$$
$$+\int_0^1(H_u^{k_1}H_u^{k_2k_3}+H_u^{k_2}H_u^{k_1k_3}+H_u^{k_3}H_u^{k_1k_2})du.$$

- $1\le k_1,k_2,k_3\le s-1,\ k$ – even (Other cases are treated similarly.) (we assume that $k_3>k_1,k_2$)

$$\mathcal{Z}_{k_1k_2k_3}=-\frac{H_u^{k_1}H_u^{k_2}H_u^{k_3}}{(H_u^0)'}\Big|_{x=1}+\int_0^1 H_u^{k_3}H_u^{k_1k_2}dx$$

where prime denotes the derivative over x. Thus to satisfy fusion rules one needs the following to hold

$$\int_0^1 H_u^{k_3}H_u^{k_1k_2}dx=\begin{cases}0, & \text{if}\quad k_3\le k_1+k_2\\ \frac{1}{p}, & \text{if}\quad k_3>k_1+k_2\end{cases}$$

This determines ($P_k^{(a,b)}$ - Jacobi polynomial)

$$H_u^{k_1k_2}=\frac{1}{p}\sum_{k=0}^{\frac{s-k_1-k_2-\alpha-2}{2}}(6k+4\alpha-3)x^{2(\alpha-1)}P_k^{(0,\frac{2}{3}(2\alpha-3))},\qquad 1\le k_1,k_2\le s-1$$

- The quantity that doesn't depend on the normalization of the operators is

$$\frac{(Z_{k_1 k_2 k_3})^2 Z_0}{\prod_{i=1}^{3} Z_{k_i k_i}} = \frac{\prod_{i=1}^{3} |p - k_i q|}{p(p+q)(p-q)}$$

where $p = 3s + \alpha, q = 3$.

- This perfectly agrees with direct calculations in Minimal Liouville gravity.

- Direct calculation then gives for even k_i

$$Z_{k_1k_2k_3k_4} = Z^{(0)}_{k_1k_2k_3k_4} + Z^{(I)}_{k_1k_2k_3k_4}$$

where the first term will reproduce the expression from Minimal gravity and the second ensure the fusion rules

$$Z^{(0)}_{k_1k_2k_3k_4} = \left(-\frac{H_u''}{(H_u')^3} + \frac{\sum_{i=1}^4 (H_u^{k_i})'}{(H_u')^2} - \frac{\sum_{i<j} H_u^{k_ik_j}}{H_u'} \right) \Bigg|_{x=1} + \\ + \int_0^1 dx (H_u^{k_1k_2} H_u^{k_3k_4} + H_u^{k_1k_3} H_u^{k_2k_4} + H_u^{k_1k_4} H_u^{k_2k_3})$$

$$Z^{(I)}_{k_1k_2k_3k_4} = \int_0^1 dx (H_u^{k_1k_2k_3} H_u^{k_4} + H_u^{k_1k_2k_4} H_u^{k_3} + H_u^{k_1k_3k_4} H_u^{k_2} + H_u^{k_2k_3k_4} H_u^{k_1} + H_u^0 H_u^{k_1k_2k_3k_4})$$

where prime means the derivative over x. We assume that $k_1 \leq k_2 \leq k_3 \leq k_4$

- $1 \leq k_1, k_2, k_3, k_4 \leq s-1$ (Other cases are treated similarly) The fusion rules demand

$$\int_0^1 dx H_u^{k_1k_2k_3} H_u^{k_4} = \begin{cases} 0, & k_4 \leq k_1 + k_2 + k_3 \\ -Z^{(0)}_{k_1k_2k_3k_4}, & k_4 > k_1 + k_2 + k_3 \end{cases}$$

- The last determines

$$H_u^{k_1 k_2 k_3} = \frac{1}{p^2} \sum_{k=0}^{\frac{s-k_1-k_2-k_3-\alpha-4}{2}} c_k x^{2(\alpha-1)} P_k^{(0,\frac{2}{3}(2\alpha-3))}(y)$$

where

$$c_k = \frac{3}{4}(6k+4\alpha-3)(2k+\sum_{i=1}^{3} k_i - s + \alpha + 2)(2k - \sum_{i=1}^{3} k_i - 2s - 2\alpha - 12)$$

- A reasonable quantity to evaluate here is again

$$\frac{(\mathcal{Z}_{k_1 k_2 k_3 k_4} \mathcal{Z}_0)^2}{\prod_{i=1}^{4} \mathcal{Z}_{k_i k_i}}$$

- Using properties of Jacobi polynomials one gets precisely the expression from minimal gravity.

- We argued that the partition function of Minimal Liouville Gravity indeed satisfies to the KdV and Douglas String equations.
- These equations together with the selection rules can be solved inspite of overdetermination of the constraints and obtained correlators agree with the results available in literature, computed by direct calculations.
- We obtain the resonance relations between KdV and Liouville observables in terms of Jacobi orthogonal polynomials.