Coset conformal field theory and instanton counting on $\mathbb{C}^2/\mathbb{Z}_p$

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Introduction

$$\text{Moduli space of } U(r) \text{ instantons on } \mathbb{C}^2/\mathbb{Z}_p$$

$$\updownarrow$$

$$\text{CFT, based on the coset } \mathcal{A}(r,p) \stackrel{\text{def}}{=} \frac{\widehat{\mathfrak{gl}}(n)_r}{\widehat{\mathfrak{gl}}(n-p)_r}$$

- This correspondence leads to the AGT relation between the partition functions of $\mathcal{N}=2$ SUSY $SU(r)$ Yang-Mills theories and correlation functions of 2d conformal field theories with the symmetry algebra $\mathcal{A}(r,p)$.
- The main aim of this work is to find nontrivial evidence in support of the conjectured correspondence between the algebra $\mathcal{A}(r,p)$ and the moduli space of $U(r)$ instantons on $\mathbb{C}^2/\mathbb{Z}_p$.
- The nontrivial fact about the moduli space $\mathcal{M}$ is that one can construct the action of some algebra $\mathcal{A}$ on the equivariant cohomologies of the moduli space. The basis in these cohomologies can be labelled by the fixed points of some abelian group (torus) acting on the moduli space. Therefore it is natural to assume the existence of the basis of geometrical origin in $\mathcal{A}$, which is in one to one correspondence with the fixed points.

We will mainly concentrate on the case of the algebra $\mathcal{A}(2,p)$. The schematic picture for this algebra is

$$\begin{tikzcd}
 & & (\mathcal{H}\times \mathrm{Vir}^{(1)})\times\ldots\times(\mathcal{H}\times \mathrm{Vir}^{(p)}) \arrow[dd, leftrightarrow] \\
\bigsqcup_N \mathcal{M}(2,N)^{\mathbb{Z}_p} \arrow[r, leftrightarrow] & \mathcal{A}(2,p) \arrow[ur] \arrow[dr] & \\
 & & \mathcal{H}^p \times \mathcal{M}(3/4)\times\ldots\times\mathcal{M}(p+1/p+2)\times\dfrac{\widehat{\mathfrak{sl}}(2)_p\times\widehat{\mathfrak{sl}}(2)_{n-p}}{\widehat{\mathfrak{sl}}(2)_n}
\end{tikzcd}$$

We find the following evidence for the conjectured correspondence between the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$ and the algebra $\mathcal{A}(r,p)$. Namely, we

- Build two realizations of the algebra $\mathcal{A}(2,p)$ and show the consistence of these two realizations and obtain the equalities between the generating functions of the fixed points of the moduli space and the characters of the representations of $\mathcal{A}(2,p)$.
- Explain the equalities between the instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_p$ from the conformal field theory point of view, assuming the existence of the bases of geometrical origin in the general case.

$\mathrm{U}(2)$ instantons on $\mathbb{C}^2$

Instantons in Yang-Mills theory, discovered by BPST, are the solutions of the self-duality equation

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}$$

and can be parametrized in terms of ADHM construction. The moduli space of the instantons with the topological charge N is given by

$$\mathcal{M}(2,N) = \left. \begin{cases} [B_1, B_2] + IJ = 0, \\ \nexists S \subset \mathrm{span}\{I_1, I_2\},\ B_{1,2}S \subset S \end{cases} \middle/ \mathrm{GL}_N \right.$$

where B_1, B_2 are $N \times N$ matrices, I – $N \times 2$ matrix and J – $2 \times N$ matrix.

Instanton partition function of $\mathcal{N} = 2$ SUSY $SU(2)$ YM on $\mathbb{C}^2$

By introduction of the Omega-deformation into the supersymmetric gauge theory, the instanton contribution becomes finite. The integral over the moduli space of instantons localizes to the points of the moduli space, which are invariant with respect to the action of the abelian group (torus)

$$B_{1,2} \mapsto t_{1,2} B_{1,2}, \ I \mapsto It, \ J \mapsto t_1 t_2 t^{-1} J,$$

where $(t_1, t_2, t) \in \mathbb{C}^* \times \mathbb{C}^* \times (\mathbb{C}^*)^2$. These fixed points can be labelled by the pairs of Young diagrams. For the pure gauge theory the instanton partition function is given by the determinants of the vector field generated by the torus in the fixed points

$$Z_{\text{inst}}^{\mathbb{C}^2}(\vec{a}, \epsilon_1, \epsilon_2 | \Lambda) = \sum_{\vec{Y}} \frac{\Lambda^{4|\vec{Y}|}}{\det v|_{P_{\vec{Y}}}}.$$

The determinant of the vector field $v = (\epsilon_1, \epsilon_2, \vec{a})$

$$\det v|_{P_{\vec{Y}}} = \prod_{i,j=1}^{2} \prod_{s \in Y_i} E_{Y_i, Y_j}(a_i - a_j | s)(\epsilon_1 + \epsilon_2 - E_{Y_i, Y_j}(a_i - a_j | s)),$$

where

$$E_{Y,W}(x|s) = x - l_W(s)\epsilon_1 + (a_Y(s) + 1)\epsilon_2.$$

$U(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_p$

- In the present work we consider the $U(2)$ instantons on $\mathbb{C}^2/\mathbb{Z}_p$. The abelian group $\mathbb{Z}_p$ acts in $\mathbb{C}^2$ in the following way

$$(z_1, z_2) \to (\omega z_1, \omega^{-1} z_2), \quad \omega^p = 1.$$

  This means, that we take into account only the solutions of the self-duality equation satisfying the condition

$$A_\mu(z_1, z_2) = A_\mu(\omega z_1, \omega^{-1} z_2).$$

- First compactification. $\mathbb{C}^2/\mathbb{Z}_p$ is not a smooth manifold, and we consider the manifold with resolved singularity at the origin $X_p = \widetilde{\mathbb{C}^2/\mathbb{Z}_p}$, which is called ALE space. One can construct the instanton moduli space $\bigsqcup_N \mathcal{M}(X_p, 2, N)$ over this manifold. The fixed points of the torus group on this manifold are enumerated by the pair of pairs of Young diagrams and the vector with $p-1$ integer components.
- Second compactification. The action of $\mathbb{Z}_p$ can be lifted to the moduli space of $U(2)$ instantons on $\mathbb{C}^2$, which is denoted by $\bigsqcup_N \mathcal{M}(2, N)$ (N is the topological charge). We regard to the set of $\mathbb{Z}_p$-invariant points on $\mathcal{M}(2, N)$ as $\mathcal{M}(2, N)^{\mathbb{Z}_p}$. In what follows we will call $\bigsqcup_N \mathcal{M}(2, N)^{\mathbb{Z}_p}$ the moduli space of instantons on $\mathbb{C}^2/\mathbb{Z}_p$. It is convenient to enumerate the fixed points of the torus action on the moduli space $\bigsqcup_N \mathcal{M}(r, N)^{\mathbb{Z}_p}$ by the pairs of the Young diagrams colored in p colors.

Instanton partition function of $\mathcal{N}=2$ SUSY $SU(2)$ YM on $\mathbb{C}^2/\mathbb{Z}_p$. First compactification

One approach to the calculation of the instanton partition function on $\mathbb{C}^2/\mathbb{Z}_p$ is the integration over the moduli space of instantons $\bigsqcup_N \mathcal{M}(X_p, 2, N)$ on the resolved space $X_p = \widetilde{\mathbb{C}^2/\mathbb{Z}_p}$. The fixed points of the torus are labelled by the pair of pairs of Young diagrams and $p-1$ integer numbers. The partition function in this approach

$$Z^{p,s}_{\text{inst}}(\vec{a}, \epsilon_1, \epsilon_2|\Lambda) = \sum_{n_1,\dots,n_p\in\mathbb{Z}} \frac{\Lambda^{(n_i+d_i^s)C_{ij}(n_j+d_j^s)}}{l^{\text{vec}}_{p,s}(a, n_1, \dots, n_p)} \prod_{\sigma=1}^{p} Z^{\mathbb{C}^2}_{\text{inst}}(\vec{a}^{(\sigma)}_s, \epsilon_1^{(\sigma)}, \epsilon_2^{(\sigma)}|\Lambda),$$

where $\vec{a} = (a, -a)$, $\vec{a}^{(\sigma)}_s = (a^{(\sigma)}_s, -a^{(\sigma)}_s)$, and $a^{(\sigma)}_s = a + (n_{\sigma+1} + d^s_{\sigma+1})\epsilon_1^{(\sigma)} + (n_\sigma + d^s_\sigma)\epsilon_2^{(\sigma)}$, and regularization parameters are $\epsilon_1^{(\sigma)} = (p-\sigma)\epsilon_1 - \sigma\epsilon_2$, $\epsilon_2^{(\sigma)} = (\sigma+1-p)\epsilon_1 + (\sigma+1)\epsilon_2$. The shifts d^s_σ are given by the formula

$$d^s_\sigma = \begin{cases} \frac{1}{p}\sigma(p-s), & \text{if} \quad \sigma \leqslant s \\ \frac{1}{p}s(p-\sigma), & \text{if} \quad \sigma > s \end{cases}, \quad s = 0, .., p-1, \quad \sigma = 1, \dots, p.$$

And C_{ij} is the $(p-1)\times(p-1)$ Cartan matrix of the simple Lie algebra A_{p-1}.

Instanton partition function of $\mathcal{N}=2$ SUSY $SU(2)$ YM on $\mathbb{C}^2/\mathbb{Z}_p$. Second compactification

In the instanton partition function corresponding to this compactification of the moduli space we take the sum only over the Young diagrams with particular coloring, and also count only the special boxes of these Young diagrams. The sum is taken over the set $\lozenge$ of the pairs of Young diagrams (Y_1, Y_2):

$$\lozenge = \{(Y_1, Y_2) | \begin{array}{|c|c|c|c|}\hline r_1 & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{3}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|c|}\hline r_2 & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}, \sharp(\boxed{m}) - \sharp(\boxed{0}) = k_m\},$$

where the box in Y_1 with the coordinates (i, j) has the color $r_1 + i - j \bmod p$ and the box (i, j) in Y_2 has the color $r_2 + i - j \bmod p$ and $\sharp(\boxed{m}), \sharp(\boxed{0})$ – the numbers of the boxes in (Y_1, Y_2) with m and 0 color respectively.
The instanton partition function is given by

$$Z_{r_1,r_2}(k_1, ..., k_{p-1}|\vec{a}, \epsilon_1, \epsilon_2|\Lambda) =$$

$$= \sum_{(Y_1,Y_2)\in\lozenge} \Lambda^{\frac{|Y_1|+|Y_2|}{p}} \prod_{i,j=1}^{2} \prod_{s\in Y_i}^{\prime} \frac{1}{E_{Y_i,Y_j}(s|a_i - a_j)(\epsilon_1 + \epsilon_2 - E_{Y_i,Y_j}(s|a_i - a_j))},$$

where the product goes only through $s \in Y_i$ that satisfy
$l_{Y_j}(s) + a_{Y_i}(s) + 1 \equiv r_j - r_i \bmod p$.

Counting of the torus fixed points of the moduli space of instantons

The generating function of the pair of colored Young diagrams can be constructed from the generating function of one coloured Young diagram. Let r be the color of the corner cell, then for $r = 0, ..., p-1$ it is defined as

$$\chi_r(k_1, ..., k_{p-1}|q) \stackrel{\text{def}}{=} \sum_{Y \in \Diamond} q^{\frac{|Y|}{p}},$$

where $|Y|$ is the number of the boxes in the Young diagram Y, and $\Diamond$ is the set of Young diagrams with particular coloring

$$\Diamond = \{Y | \begin{array}{|c|c|c|c|} \hline r & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}, \sharp(\boxed{m}) - \sharp(\boxed{0}) = k_m\},$$

where the box with coordinates (i, j) has $r + i - j \bmod p$ color and

$$\sharp(\boxed{m})\sharp(\boxed{0}) \; - \; \text{the number of the boxes with } m \text{ and } 0 \text{ color respectively.}$$

One can get for the generating function of one colored Young diagram (here we imply $k_0 = k_p = 0$)

$$\chi_r(k_1, ..., k_{p-1}|q) = q^{\sum_{i=1}^{p-1}(k_i^2 + \frac{k_i}{p} - k_i k_{i+1}) - k_r} \cdot (\chi_B(q))^p.$$

The generating function of the pair of Young diagrams

$$\chi_{r_1,r_2}(k_1,...,k_{p-1}|q)=\sum_{\substack{m_i+n_i=k_i\\ i=1,...,p-1}}\chi_{r_1}(m_1,...,m_{p-1}|q)\chi_{r_2}(n_1,...,n_{p-1}|q).$$

Then one obtains

$$\chi_{r_1,r_2}(k_1,...,k_{p-1}|q)=(\chi_B(q))^{2p}\cdot$$
$$\cdot\sum_{m_i\in\mathbb{Z}}q^{\frac{1}{2}\sum_{i=1}^{p-1}((2m_i-k_i)^2-(2m_i-k_i)(2m_{i+1}-k_{i+1})+k_i^2-k_ik_{i+1}+\frac{2k_i}{p})-m_{r_1}+m_{r_2}-k_{r_2}}.$$

We see that the form of the obtained generating function for the pair of colored Young diagrams is not as simple as the generating function for one colored Young diagram, which is simply proportional to $\chi_B^p(q)$. However, in what follows we show that the generating functions are divided into finite number of classes with the generating functions being proportional to each other within each class.

Counting of the non-equivalent generating functions

We conclude that the generating functions of the pair of colored Young diagrams have the following symmetries:

- The invariance under the transformation $k_m \to k_m + 2$:

$$\chi_{r_1,r_2}(k_1, \ldots, k_m + 2, \ldots, k_{p-1}|q) = \\ = q^{2k_m - k_{m+1} - k_{m-1} + \frac{2}{p} + \delta_{m,r_1} + \delta_{m,r_2}} \chi_{r_1,r_2}(k_1, \ldots, k_{p-1}|q),$$

  where $\delta_{m,n}$ is the Kronecker delta.

- The invariance under the twist $r_1 \leftrightarrow r_2$:

$$\chi_{r_1,r_2}(k_1, \ldots, k_{p-1}|q) = \chi_{r_2,r_1}(k_1, \ldots, k_{p-1}|q).$$

- The invariance under the change $r_1, r_2 \to r_1 + 1, r_2 + 1$:

$$\chi_{r_1+1,r_2+1}(k_1, \ldots, k_{p-1}|q) = \\ = q^{k_{r_1} - k_{r_1+1} - \frac{r_2 - r_1}{p}} \chi_{r_1,r_2}(k_1, \ldots, k_{r_1+1} + 1, \ldots, k_{r_2} + 1, \ldots, k_{p-1}|q).$$

  where we assume that $r_1 \leqslant r_2$.

Therefore, every generating function of the pair of colored Young diagrams is equivalent to one of the generating functions of the form

$$\chi_{0,s}(k_1, ..., k_{p-1}|q),$$

where $s = 0, 1, ..., p-1$ all k_m's are equal to 0 or 1.
The properties of the generating functions allow us to prove that the following symmetries take place

$$\begin{aligned}
\chi_{0,s}(..., 0, \overset{j}{0}, 1, ...|q) &= q^{-\frac{1}{p}} \chi_{0,s}(..., 0, \overset{j}{1}, 1, ...|q), \quad j \neq s \\
\chi_{0,s}(..., 1, \overset{j}{0}, 0, ...|q) &= q^{-\frac{1}{p}} \chi_{0,s}(..., 1, \overset{j}{1}, 0, ...|q), \quad j \neq s \\
\chi_{0,s}(..., 0, \overset{s}{0}, 0, ...|q) &= q^{-\frac{1}{p}} \chi_{0,s}(..., 0, \overset{s}{1}, 0, ...|q), \\
\chi_{0,s}(..., 1, \overset{s}{0}, 1, ...|q) &= q^{1-\frac{1}{p}} \chi_{0,s}(..., 1, \overset{s}{1}, 1, ...|q),
\end{aligned}$$

where the dots ... stand for an arbitrary combinations of zeros and unities.

It can be shown that for each $s = 0, \ldots, p-1$ the generating functions are divided into $[s/2] + [(p-s)/2] + 1$ classes of equivalence. The first class of equivalence is represented by the generating functions equivalent to

$$\chi_{0,s}(0, ..., 0|q),$$

and its cardinality is equal to C_p^s. For each of the next $[s/2]$ classes of equivalence it is convenient to choose the representative

$$\chi_{0,s}(0, ..., \overset{s-2l+1}{0, 1, 0}, 1, 0, ..., 1, 0, 1, \overset{s}{0}, 0, ..., 0),$$

with l taking integer values from 1 to $[s/2]$. The cardinality of the equivalence class with given l is C_p^{s-2l}. And for each of the last $[(p-s)/2]$ classes of equivalence we choose the representative

$$\chi_{0,s}(0, ..., 0, \overset{s}{0}, 1, 0, 1, 0, ..., 1, \overset{s+2n-1}{0, 1, 0}, ..., 0),$$

where n takes the values from 1 to $[(p-s)/2]$.

First realization of the algebra $\mathcal{A}(2, p)$

Application of the level-rank duality gives

$$\mathcal{A}(2, p) \supset \mathcal{H}^p \times \mathrm{Vir}^{(1)} \times \ldots \times \mathrm{Vir}^{(p)}$$

where $\mathrm{Vir}^{(\sigma)}$ are the Virasoro algebras with the following central charges

$$c_\sigma = 1 + \frac{3(n-\sigma)}{n-\sigma+2} - \frac{3(n-\sigma+1)}{n-\sigma+3} = 1 + 6(Q_\sigma)^2,$$

where $Q_\sigma = b_\sigma + b_\sigma^{-1}$ and $b_\sigma^2 = -\frac{n-\sigma+3}{n-\sigma+2}$ is a parametrization.
It is easy to check that the parameters b_σ satisfy the following relations

$$b_\sigma^2 + b_{\sigma+1}^{-2} = -2, \quad \sigma = 1, \ldots, p-1.$$

In addition to p stress-energy tensors $T^{(\sigma)}$, which generate the Virasoro symmetry with the central charges c_σ, we form the set of the $p-1$ holomorphic currents:

$$J^{(\sigma)}(z) \stackrel{\mathrm{def}}{=} V_{1,2}^{(\sigma)}(z) V_{2,1}^{(\sigma+1)}(z), \quad \sigma = 1, \ldots, p-1,$$

where $V_{m,n}$ is degenerate field. Because of the relation for b_σ's, the left conformal dimension of the current $J^{(\sigma)}(z)$

$$\Delta_{J^{(\sigma)}} = \Delta_{1,2}^{(\sigma)} + \Delta_{2,1}^{(\sigma+1)} = \frac{1}{2}.$$

Representations of the product of p Models

Consider the OPE of the currents $J^{(\sigma)}(z)$ with the state $V^{(1)}_{\lambda_1}\dots V^{(p)}_{\lambda_p}$. From the fusion rules it follows

$$J^{(\sigma)}(z)V^{(1)}_{\lambda_1}(0)\dots V^{(p)}_{\lambda_p}(0) = \sum_{m_\sigma, m_{\sigma+1}=\pm 1} z^{m_\sigma\lambda_\sigma b_\sigma + m_{\sigma+1}\lambda_{\sigma+1}b^{-1}_{\sigma+1}} \times$$

$$\times\, C^{(\sigma)}(m_\sigma, m_{\sigma+1}; \lambda_1, \dots, \lambda_p)\left[V^{(1)}_{\lambda_1}\dots V^{(\sigma)}_{\lambda_\sigma + \frac{m_\sigma b_\sigma}{2}} V^{(\sigma+1)}_{\lambda_{\sigma+1}+\frac{m_{\sigma+1}}{2b_{\sigma+1}}}\dots V^{(p)}_{\lambda_p}\right],$$

where $C^{(\sigma)}(m_\sigma, m_{\sigma+1}; \lambda_1, \dots, \lambda_p)$ are the structure constants. To reach locality we have to make the projection and keep only two terms in the sum with $m_\sigma = m_{\sigma+1} = \pm 1$ and also impose condition $\lambda_\sigma b_\sigma + \lambda_{\sigma+1}b^{-1}_{\sigma+1} \in \mathbb{Z}$ or $\mathbb{Z} + 1/2$.

$$J^{(\sigma)}(z)V^{(1)}_{\lambda_1}(0)\dots V^{(p)}_{\lambda_p}(0) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} z^{n-\frac{1}{2}} J^{(\sigma)}_n V^{(1)}_{\lambda_1}(0)\dots V^{(p)}_{\lambda_p}(0),$$

$$J^{(\sigma)}(z)V^{(1)}_{\lambda_1}(0)\dots V^{(p)}_{\lambda_p}(0) = \sum_{n\in\mathbb{Z}} z^{n-\frac{1}{2}} J^{(\sigma)}_n V^{(1)}_{\lambda_1}(0)\dots V^{(p)}_{\lambda_p}(0).$$

The second requirement is that the state is primary for the currents $J^{(\sigma)}$, i.e. it's annihilated by the modes of all currents $J^{(\sigma)}(z)$ with positive numbers

$$J_n^{(\sigma)} V_{\lambda_1}^{(1)}(0) \ldots V_{\lambda_p}^{(p)}(0) = 0, \; n > 0, \; \sigma = 1, \ldots, p-1.$$

This condition leads us to the following relation for the Liouville momenta $\lambda_1, \lambda_2, \ldots, \lambda_p$:

$$\lambda_\sigma b_\sigma + \lambda_{\sigma+1} b_{\sigma+1}^{-1} = 0,$$

if the modes of the σ-th current $J^{(\sigma)}(z)$ are half-integer and

$$\lambda_\sigma b_\sigma + \lambda_{\sigma+1} b_{\sigma+1}^{-1} = \pm \frac{1}{2},$$

if these modes are integer.

- NS-type representation.

$$\lambda^0_\sigma b_\sigma + \lambda^0_{\sigma+1} b^{-1}_{\sigma+1} = 0$$

The primary state with the convenient parametrization of the Liouville momenta

$$V^{(1)}_{\lambda^0_1} V^{(2)}_{\lambda^0_2} \dots V^{(p)}_{\lambda^0_p}, \quad \lambda^0_\sigma = \frac{\lambda}{\sqrt{(\sigma Q - pb)(pb - (\sigma-1)Q)}}.$$

- R-type representations.

$$\lambda^s_\sigma b_\sigma + \lambda^s_{\sigma+1} b^{-1}_{\sigma+1} = \begin{cases} 0, & \sigma \neq s, \\ -\frac{1}{2}, & \sigma = s. \end{cases}$$

Introducing the new variables d^s_σ

$$d^s_\sigma = \begin{cases} \frac{1}{p}\sigma(p-s), & \text{if} \quad \sigma \leqslant s \\ \frac{1}{p}s(p-\sigma), & \text{if} \quad \sigma > s \end{cases} , \quad s = 0, .., p-1, \quad \sigma = 1, ..., p$$

we are able to express λ^s_σ in terms of λ^0_σ

$$\lambda^s_\sigma = \lambda^0_\sigma + d^s_{\sigma-1} \frac{b^{-1}_\sigma}{2} + d^s_\sigma \frac{b_\sigma}{2}.$$

Thus, we obtain the primary state in the s-th R representation

$$V^{(1)}_{\lambda^s_1} V^{(2)}_{\lambda^s_2} \dots V^{(p)}_{\lambda^s_p}.$$

Characters of the representations of p Models

Therefore, the s-th representation of the first realization of $\mathcal{A}(2,p)$

$$\pi^{\mathbf{1}}_{p,s} \stackrel{\text{def}}{=} \bigoplus_{n_1,\dots,n_{p-1}\in\mathbb{Z}} \left[V^{(1)}_{\lambda^s_1+n_1\frac{b_1}{2}}\right]\times\left[V^{(2)}_{\lambda^s_2+n_1\frac{b_2^{-1}}{2}+n_2\frac{b_2}{2}}\right]\times\dots\times\left[V^{(p)}_{\lambda^s_p+n_{p-1}\frac{b_p^{-1}}{2}}\right],$$

where $s=0,\dots,p-1$ and square brackets denote the Virasoro module.
The character of the representation $\pi^{\mathbf{1}}_{p,s}$

$$\chi^s_p(q)=\mathrm{tr}\left\{q^{\sum_{\sigma=1}^p L_0^{(\sigma)}}\right\}_{\pi^{\mathbf{1}}_{p,s}}.$$

$$\chi^s_p(q)=(\chi_B(q))^p\sum_{\substack{n_1,\dots,n_{p-1}\in\mathbb{Z}\\ n_0=n_p=0}} q^{\sum_{\sigma=1}^p\Delta^{(\sigma)}(\lambda^s_\sigma+n_{\sigma-1}\frac{b_\sigma^{-1}}{2}+n_\sigma\frac{b_\sigma}{2})}.$$

Calculating the sum of conformal dimensions one gets

$$\chi^s_p(q)=q^{\Delta_{p,s}(\lambda)}(\chi_B(q))^p\sum_{\substack{n_1,\dots,n_{p-1}\in\mathbb{Z}\\ n_0=n_p=0}} q^{\frac{1}{2}\sum_{\sigma=1}^{p-1}(n_\sigma^2-n_\sigma n_{\sigma+1})+\frac{1}{2}n_s},$$

where $\Delta_{p,s}(\lambda)=(Q^2/4-\lambda^2)/p+s(p-s)/(4p)$.

Comparison with the generating functions of colored Young diagrams

One can show now the coincidence of the generating functions of the fixed points on the moduli space and the characters of the first realization of $\mathcal{A}(2,p)$:

$$q^{-\Delta_{p,s}(\lambda)}(\chi_B(q))^p\chi_p^s(q) =$$
$$= \sum_{k_1,\ldots,k_{p-1}=0}^{1} q^{-\frac{1}{2}\sum_{i=1}^{p-1}(k_i^2-k_ik_{i+1}+\frac{2k_i}{p})+\frac{k_s}{2}}\chi_{0,s}(k_1,\ldots,k_{p-1}|q),$$

Applying the symmetries of the generating functions

$$q^{-\Delta_{p,s}(\lambda)}(\chi_B(q))^p\chi_p^s(q) = C_p^s\chi_{0,s}(0,\ldots,0|q)+$$
$$+\sum_{n=1}^{[s/2]} C_p^{s-2n}q^{-\frac{n}{2}\left(1+\frac{2}{p}\right)}\chi_{0,s}(0,\ldots,0,\overset{s-2n+1}{0,1,0},\ldots,1,0,1,\overset{s}{0},0,\ldots,0|q)+$$
$$+\sum_{n=1}^{[(p-s)/2]} C_p^{s+2n}q^{-\frac{n}{2}\left(1+\frac{2}{p}\right)}\chi_{0,s}(0,\ldots,0,\overset{s}{0},1,0,1,\ldots,\overset{s+2n-1}{0,1,0},0,\ldots,0|q).$$

Below we illustrate the obtained identity by listing the examples for the $p=2,3$.

$$q^{-\Delta_{2,0}(\lambda)}\chi_B^2(q)\chi_2^0(q) = \chi_{0,0}(0|q)+q^{-1}\chi_{0,0}(1|q),$$
$$q^{-\Delta_{2,1}(\lambda)}\chi_B^2(q)\chi_2^1(q) = 2\chi_{0,1}(0|q),$$
$$q^{-\Delta_{3,0}(\lambda)}\chi_B^3(q)\chi_3^0(q) = \chi_{0,0}(0,0|q)+3q^{-\frac{5}{6}}\chi_{0,0}(1,0|q),$$
$$q^{-\Delta_{3,1}(\lambda)}\chi_B^3(q)\chi_3^1(q) = 3\chi_{0,1}(0,0|q)+q^{-\frac{5}{6}}\chi_{0,1}(0,1|q).$$

Second realization of the algebra $\mathcal{A}(2, p)$

- Applying the level-rank duality in the other way, we get

$$\mathcal{A}(2, p) \supset \mathcal{H}^p \times \mathcal{M}(3/4) \times ... \times \mathcal{M}(p+1/p+2) \times \frac{\widehat{\mathfrak{sl}}(2)_p \times \widehat{\mathfrak{sl}}(2)_{n-p}}{\widehat{\mathfrak{sl}}(2)_n}.$$

- Further we will show that the character of a certain sum of the representations of the right hand side coincides with the character of the representation of the first realization of $\mathcal{A}(2, p)$, which means that two realizations of $\mathcal{A}(2, p)$ are consistent.

- Then, automatically all characters will be equal to the sum of the generating functions of the pairs of colored Young diagrams.

Representations of the coset $\widehat{\mathfrak{sl}}(2)_p \times \widehat{\mathfrak{sl}}(2)_{n-p} / \widehat{\mathfrak{sl}}(2)_n$

The representation of the numerator $\pi_{p,\frac{m}{2}} \times \pi_{n-p,j}$ is decomposed into the sum of the irreducible representations of the product of the denominator and coset itself:

$$\pi_{p,\frac{m}{2}} \otimes \pi_{n-p,j} = \bigoplus_{\substack{s\in\mathbb{Z} \\ m-s=0 \bmod 2}} \pi_{n,j+\frac{s}{2}} \otimes V_s^m(p,j),$$

We pass from the parameters n and j to the b and λ as follows

$$b^2 = -\frac{n+2}{n-p+2}, \quad j = \frac{1}{Q}\left(\mu - \frac{Q}{2} - \frac{s}{2b}\right), \quad Q = b + b^{-1}.$$

The dimension of the highest weight representation and its character are

$$\Delta_s^m(\mu) = \begin{cases} \frac{1}{p}\left(\frac{Q^2}{4} - \mu^2\right) + \frac{s(p-s)}{2p(p+2)} + \frac{(m-s)(m+s+2)}{4(p+2)}, & m \geqslant s \\ \frac{1}{p}\left(\frac{Q^2}{4} - \mu^2\right) + \frac{s(p-s)}{2p(p+2)} + \frac{(s-m)(2p-m-s+2)}{4(p+2)}, & m < s \end{cases}$$

$$c_s^m(q) = q^{\Delta_s^m(\mu)} \chi_B^3(q) \sum_{r,l=0}^{\infty} (-1)^{r+l} q^{\frac{l(l+1)}{2} + \frac{r(r+1)}{2} + rl(p+1)} \times$$

$$\times \left(q^{l\frac{m-s}{2} + r\frac{m+s}{2}} - q^{p+1-m+l(p+1-\frac{m-s}{2})+r(p+1-\frac{m+s}{2})}\right).$$

where $0 \leqslant m, s \leqslant p,\ m-s = 0 \bmod 2$.

Product of consecutive Minimal Models

Consider the product of $p-1$ Minimal models highest-weight states $\phi^{(3)}_{1k_1}\times\phi^{(4)}_{k_1k_2}\times\ldots\times\phi^{(p+1)}_{k_{p-2}n}$, where k_i runs from 1 to $i+2$. This composite highest weight state has a dimension (hereafter we imply $k_0=1, k_{p-1}=n$)

$$h_n(k_1,\ldots,k_{p-2})=\sum_{i=1}^{p-1}h^{(i+2)}_{k_{i-1}k_i}=\frac{(n^2-1)(p+1)}{4(p+2)}+\frac{1}{2}\sum_{i=0}^{p-2}(k_i^2-k_ik_{i+1}).$$

The irreducible representation which is built from this composite highest-weight state is $M^{(3)}_{1,k_1}\times M^{(4)}_{k_1,k_2}\times\ldots.\times M^{(p+1)}_{k_{p-2},n}$. Introduce a convenient notation for the character of this representation:

$$\mathrm{ch}_n(q)\overset{\mathrm{def}}{=}\sum_{\substack{\{k_1,..,k_{p-2}\}\\1\leqslant k_i\leqslant i+2}}\prod_{i=1}^{p-1}\chi^{(i+2)}_{k_{i-1}k_i}(q).$$

Comparison with the generating functions of Young diagrams

We consider the following representations of the coset and consecutive product of Minimal models

$$[\Psi_s^m(\mu)] \times \sum_{\substack{\{k_1,..,k_{p-2}\} \\ 1\leqslant k_i \leqslant i+2}} M^{(3)}_{1,k_1} \times M^{(4)}_{k_1,k_2} \times .... \times M^{(p+1)}_{k_{p-2},n},$$

where $1 \leqslant n \leqslant p+1$, $0 \leqslant m, s \leqslant p$ with $m - s = 0 \bmod 2$. Therefore, the dimensions of the representations of the coset and minimal models with the parameters n, m, and s should differ only by integer or half-integer number, thus we get the following Diophantine equation:

$$\Delta_s^m(\mu) + h_n(k_1, ..., k_{p-2}) - \Delta_{p,s}(\lambda) \in \mathbb{Z}/2.$$

Identifying $\mu = \lambda$, and because $-p \leqslant m - n + 1 \leqslant p$ and $2 \leqslant m + n + 1 \leqslant p + 2$, there exist two possibilities

$$n = m + 1, \quad n = p - m + 1.$$

Now one can find the following equality of characters of representations

$$\sum_{\substack{0\leqslant m \leqslant p \\ m-s=0 \bmod 2}} c_s^m(q)(\mathrm{ch}_{m+1}(q) + \mathrm{ch}_{p-m+1}(q)) = \chi_p^s(q).$$

We have checked the equality for the cases $p = 2, ..., 8$.

Equality of the partition functions

- We check the same equality for the instanton partition functions in the different compactifications, as for the characters of the first realization of $\mathcal{A}(2,p)$

$$Z_{\text{inst}}^{p,s}(\vec{a},\epsilon_1,\epsilon_2|\Lambda)=$$
$$=\sum_{k_1,\ldots,k_{p-1}=0}^{1}\Lambda^{-\frac{1}{2}\sum_{i=1}^{p-1}(k_i^2-k_ik_{i+1}+\frac{2k_i}{p})+\frac{k_s}{2}}Z_{0,s}(k_1,\ldots,k_{p-1}|\vec{a},\epsilon_1,\epsilon_2|\Lambda).$$

- It is interesting, that the following relation, which implies the same symmetries for the instanton partition functions as for the generating functions of Young diagrams, also holds

$$Z_{\text{inst}}^{p,s}(\vec{a},\epsilon_1,\epsilon_2|\Lambda)=C_p^sZ_{0,s}(0,\ldots,0|\vec{a},\epsilon_1,\epsilon_2|\Lambda)+$$
$$+\sum_{n=1}^{[s/2]}C_p^{s-2n}q^{-\frac{n}{2}\left(1+\frac{2}{p}\right)}Z_{0,s}(0,\ldots,0,\overset{s-2n+1}{0,1,0},\ldots,1,0,1,\overset{s}{0},0,\ldots,0|\vec{a},\epsilon_1,\epsilon_2|\Lambda)+$$
$$+\sum_{n=1}^{[(p-s)/2]}C_p^{s+2n}q^{-\frac{n}{2}\left(1+\frac{2}{p}\right)}Z_{0,s}(0,\ldots,0,\overset{s}{0},1,0,1,\ldots,\overset{s+2n-1}{0,1,0},0,\ldots,0|\vec{a},\epsilon_1,\epsilon_2|\Lambda),$$

- These identities can be explained by using the representation of the instanton partition function as the norm of the Whittaker vector

$$Z_{\text{pure}}=\langle W|W\rangle,$$

introducing the complete sets of states in each basis of geometrical origin corresponding to different compactifications.

Conclusions

- We showed, that the generating functions of the pairs of colored Young diagrams, which label the fixed points of the moduli space, can be divided into the finite number of the classes of equivalence.
- Two realizations of the algebra $\mathcal{A}(2,\mathrm{p})$ were constructed and the characters of the representations in each realization were calculated.
- It was shown that these two realizations are consistent. We also proved that the character of the representation of the first realization of $\mathcal{A}(2,\mathrm{p})$ equals to the certain sum of the generating functions of colored Young diagrams.
- The obtained equalities allow to explain the equalities between the instanton partition functions on $\mathbb{C}^2/\mathbb{Z}_\mathrm{p}$, corresponding to the different compactifications of the moduli space.