The Cauchy problem for the Pavlov equation

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Landau Days 2014, Session: In memory of Sergey Manakov ,
June 25, 2014.

Two important classes of intregrable hierarchies:

- Equations with non-zero dispersion
- Dispersionless systems (hydrodynamical-type equations).

Equations with non-zero dispersion are studied much better: Korteveg-de Vries equation, Nonlinear Schrödinger equation, Sine-Gordon equation, Kadomtsev-Petviashvili equation ...
The Lax pair contains higher-order differential operators.
Dressing method, Darbough transformations, soliton solutions, finite-gap integration.

Dispersionless systems

Dispersionless integrable systems:

- They have no soliton solutions
- They may have wave breaking.
- They may have arbitrary many spatial variables.

The study of 1+1 dispersionless systems as completely integrable sytems was started in the middle of 1980's.
Dubrovin B.A., Novikov S.P., "Hamiltonian formalism of one-dimensional systems of the hydrodynamic type and the Bogolyubov-Whitham averaging method", Soviet Math. Dokl. 27,(1983), 665-669.
Tsarev S.P., "On Poisson bracket and one-dimensional systems of hydrodynamic type, Soviet Math. Dokl. 31 (1985), 488-491.
Different integration methods - generalized Hodograph transformation.

Lax pairs for multidimensional dispersionless systems based on vector fields:
V. E. Zakharov and A. B. Shabat, "Integration of nonlinear equations of mathematical physics by the method of inverse scattering. II", Functional Anal. Appl. 13, (1979), 166-174.

A series of papers by S.V. Manakov, P.M. Santini – how to develop an analog of the dressing method for dispersionless systems with more than one spatial variables?
S.V. Manakov and P. M. Santini "Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation", Physics Letters A 359 (2006) 613-619. http://arXiv:nlin.SI/0604017.
S. V. Manakov and P. M. Santini "On the solutions of the second heavenly and Pavlov equations", J. Phys. A: Math. Theor. 42 (2009) 404013 (11pp). doi: 10.1088/1751-8113/42/40/404013.

S.V. Manakov asked Santini and me to study, how to make this approach mathematically rigorous. It turns out, that development of a proper analog of spectral transform for zero dispersion case is a very non-trivial mathematical problem. In this paper we solve the Caughy problem for the mathematically simplest equation of such type – the so-called Pavlov equation.

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x, y, t),$$

assuming, that the Cauchy data

$$v(x, y, 0)$$

is sufficiently small.

1. Dispersionless Kadomtsev Petviashvili equation = Khokhlov-Zabolotskaya equation.

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \qquad x, y, t \in \mathbb{R},$$

Used in physical models:
C. C. Lin, E. Reissner, and H.S. Tsien, "On two-dimensional non-steady motion of a slender body in a compressible fluid". Journal of Mathematical Physics, 27, (1948). 220-231.
Lax representation $[\hat{L}_1, \hat{L}_2] = 0$:

$$\hat{L}_1 \equiv \partial_y + \lambda\partial_x - u_x\partial_\lambda,$$
$$\hat{L}_2 \equiv \partial_t + (\lambda^2 + u)\partial_x + (-\lambda u_x + u_y)\partial_\lambda$$

where $\lambda \in \mathbb{C}$ – is the spectral parameter.

Dispersionless Kadomtsev-Petviashvili equation

Exact solutions of dKP using algebro-geometrical methods:
I. M. Krichever, "Method of averaging for two-dimensional "integrable"equations", Funkts. Anal. Prilozh., 22:3 (1988), 37–52
The Lax representation of DKP is the quasiclassical limit of the KP Lax representation.
V. E. Zakharov "Dispersionless limit of integrable systems in 2+1 dimensions", in Singular Limits of Dispersive Waves, edited by N.M.Ercolani et al., Plenum Press, New York, 1994.
Some solutions of the DKP (as well as solutions of Whitham equation for n-phase KP averaging).
I. M. Krichever "The τ-function of the universal Witham hierarchy, matrix models and topological field theories", Comm. Pure Appl. Math. 47, 437-475 (1994).

Semiclassical version of the non-local $\bar{\partial}$ problem:
Konopelchenko, B.; Martínez Alonso, L.; Ragnisco, O. The $\bar{\partial}$-approach to the dispersionless KP hierarchy. J. Phys. A 34 (2001), no. 47, 10209–10217.
Bogdanov, L.V., Konopel'chenko, B. G.; Martines Alonso, L. The quasiclassical $\bar{\partial}$-method: generating equations for dispersionless integrable hierarchies. Theoret. and Math. Phys. 134 (2003), no. 1, 39–46,
Quasiclassical analog of the Lax pairs in these paper is non-linear.
The approach by Manakov and Santini in based on linear operators.

Some other examples from the paper:
S. V. Manakov, P. M. Santini "Solvable vector nonlinear Riemann problems, exact implicit solutions of dispersionless PDEs and wave breaking". arXiv:1011.2619 [nlin.SI]

2. The vector nonlinear PDE in $N+4$ dimensions:

$$\vec{U}_{t_1 z_2} - \vec{U}_{t_2 z_1} + \left(\vec{U}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_2} - \left(\vec{U}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{U}_{z_1} = \vec{0},$$

where $\vec{U}(t_1, t_2, z_1, z_2, \vec{x}) \in \mathbb{R}^N$, $\vec{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$ and $\nabla_{\vec{x}} = (\partial_{x_1}, .., \partial_{x_N})$,
The Lax operators are $(N+1)$ dimensional vector fields

$$\hat{L}_i = \partial_{t_i} + \lambda \partial_{z_i} + \vec{U}_{z_i} \cdot \nabla_{\vec{x}}, \quad i = 1, 2.$$

3. Its dimensional reduction, for $N = 2$:

$$\vec{U}_{tx} - \vec{U}_{zy} + \left(\vec{U}_y \cdot \nabla_{\vec{x}}\right)\vec{U}_x - \left(\vec{U}_x \cdot \nabla_{\vec{x}}\right)\vec{U}_y = \vec{0},$$
$$\vec{U} \in \mathbb{R}^2, \quad \vec{x} = (x, y), \quad \nabla_{\vec{x}} = (\partial_x, \partial_y),$$

where: $t_1 = z$, $t_2 = t$, $x_1 = x$, $x_2 = y$ and

$$\hat{L}_1 = \partial_z + \lambda\partial_x + \vec{U}_x \cdot \nabla_{\vec{x}},$$
$$\hat{L}_2 = \partial_t + \lambda\partial_y + \vec{U}_y \cdot \nabla_{\vec{x}}.$$

4. Its Hamiltonian reduction $\nabla_{\vec{x}} \cdot \vec{U} = 0$, $U_1 = \theta_y$, $U_2 = -\theta_x$ gives the celebrated second heavenly equation of Plebanski:

$$\theta_{tx} - \theta_{zy} + \theta_{xx}\theta_{yy} - \theta_{xy}^2 = 0, \quad \theta = \theta(x, y, z, t) \in \mathbb{R}, \qquad x, y, z, t \in \mathbb{R},$$

The Lax operators:

$$\hat{L}_1 \equiv \partial_z + \lambda\partial_x + \theta_{xy}\partial_x - \theta_{xx}\partial_y,$$
$$\hat{L}_2 \equiv \partial_t + \lambda\partial_y + \theta_{yy}\partial_x - \theta_{xy}\partial_y.$$

5. The two-dimensional dispersionless Toda (2ddT) equation
J. D. Finley and J. F. Plebanski "The classification of all $\mathcal{K}$ spaces admitting a Killing vector", J. Math. Phys. 20, 1938 (1979).
V. E. Zakharov "Integrable systems in multidimensional spaces", Lecture Notes in Physics, Springer-Verlag, Berlin 153 (1982), 190-216.

$$\phi_{\zeta_1\zeta_2} = \left(\mathrm{e}^{\phi_\mathrm{t}}\right)_\mathrm{t}, \quad \phi = \phi(\zeta_1, \zeta_2, \mathrm{t})$$

(or $\varphi_{\zeta_1\zeta_2} = (\mathrm{e}^{\varphi})_{\mathrm{tt}}$, $\varphi = \phi_\mathrm{t}$),
The Lax operators:
K. Takasaki and T. Takebe "SDIFF(2) hierarchy", Proceedings of the RIMS Research Project 91 "Infinite Analysis". RIMS-814, 1991.

$$\hat{\mathrm{L}}_1 = \partial_{\zeta_1} + \lambda \mathrm{e}^{\frac{\phi_\mathrm{t}}{2}} \partial_\mathrm{t} + \left(-\lambda(\mathrm{e}^{\frac{\phi_\mathrm{t}}{2}})_\mathrm{t} + \frac{\phi_{\zeta_1\mathrm{t}}}{2}\right)\lambda\partial_\lambda,$$

$$\hat{\mathrm{L}}_2 = \partial_{\zeta_2} + \lambda^{-1} \mathrm{e}^{\frac{\phi_\mathrm{t}}{2}} \partial_\mathrm{t} + \left(\lambda^{-1}(\mathrm{e}^{\frac{\phi_\mathrm{t}}{2}})_\mathrm{t} - \frac{\phi_{\zeta_2\mathrm{t}}}{2}\right)\lambda\partial_\lambda,$$

6. A system of two nonlinear PDEs in $2+1$ dimensions:

$$u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0,$$
$$v_{xt} + v_{yy} + u v_{xx} + v_x v_{xy} - v_y v_{xx} = 0,$$

with Lax operators:

$$\tilde{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x \partial_\lambda,$$
$$\tilde{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda,$$

describing a general integrable Einstein-Weyl metric
M. Dunajski, The nonlinear graviton as an integrable system, PhD Thesis, Oxford University, 1998.
M. Dunajski "An interpolating dispersionless integrable system"; J. Phys. A 41 (2008), no. 31, 315202, 9 pp. arXiv:0804.1234.

7. Its $v = 0$ reduction is dKP (Khokhlov-Zabolotskaya).

The so-called Pavlov equation:

$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \quad v = v(x,y,t) \in \mathbb{R}, \quad x,y,t \in \mathbb{R},$$

commutativity condition for the following pair of vector fields:

$$L = \partial_y + (\lambda + v_x)\partial_x,$$
$$M = \partial_t + (\lambda^2 + \lambda v_x - v_y)\partial_x,$$

where $\lambda \in \mathbb{C}$ – spectral parameter.

M. V. Pavlov "Integrable hydrodynamic chains", J. Math. Phys. 44 (2003) 4134-4156.

M. Dunajski "A class of Einstein-Weyl spaces associated to an integrable system of hydrodinamic type", J. Geom. Phys. 51 (2004), 126-137.

We show, that for "sufficiently good" Cauchy data, satisfying, in particular, the "small norm condition", the spectral transform for the Pavlov equation provides us a regular solution for all $t > 0$.
Remark. Manakov and Santini used two different formulations for the inverse spectral problem:

- The approach based on a singular integrable equation for the wave function.
- The approach based on the nonlinear Riemann-Hilbert problem.

They are not equivalent.
We us the first one.

To avoid extra technicalities, we assume that $v_0(x,y) = v(x,y,0) \in \mathbb{R}$ is smooth and has compact support:

$$v_0(x,y) = 0 \text{ outside the area } -D_x \le x \le D_x, -D_y \le x \le D_y.$$

Step 1: We construct the Jost functions and the classical scattering data. By definition, the Jost functions are solutions of:

$$L\varphi_{\pm}(x,y,\lambda) = 0, \quad L = \partial_y + (\lambda + v_x)\partial_x,$$

such that

$$\varphi_{\pm}(x,y,\lambda) \to x - \lambda y \quad \text{as} \quad y \to \pm\infty.$$

The zero eigenfunctions of L – are exactly the functions, which are constant on the characteristics, i.e. are constant on the solutions of the corresponding ODE:

$$\frac{dx}{dy} = \lambda + v_x(x,y).$$

Direct spectral transform

Consider the solutions of the following Cauchy problem:

$$\mathrm{x}(\mathrm{y}) = \mathrm{x}_0 \quad \text{при} \quad \mathrm{y} = \mathrm{y}_0.$$

We have the following asymptotic:

$$\mathrm{x}(\mathrm{y}) \to \lambda \mathrm{y} + \mathrm{x}_{\pm}(\mathrm{x}_0, \mathrm{y}_0, \lambda), \quad \mathrm{y} \to \pm\infty.$$

It is easy to see that

$$\mathrm{x}_{\pm}(\mathrm{x}_0, \mathrm{y}_0, \lambda) \to \mathrm{x}_0 - \lambda \mathrm{y}_0 \quad \text{as} \quad \mathrm{y}_0 \to \pm\infty;$$

therefore

$$\varphi_{\pm}(\mathrm{x}_0, \mathrm{y}_0, \lambda) = \mathrm{x}_{\pm}(\mathrm{x}_0, \mathrm{y}_0, \lambda).$$

The classical scattering amplitude $\sigma(\xi,\lambda)$ is defined $\xi\in\mathbb{R}$, $\lambda\in\mathbb{R}$ as the function connecting the asymptotic at $y\to+\infty$ and $y\to-\infty$

$$x_+(x_0,y_0,\lambda)=x_-(x_0,y_0,\lambda)+\sigma(x_-(x_0,y_0,\lambda),\lambda).$$

Therefore

$$\varphi_+(x,y,\lambda)\to x-\lambda y+\sigma(x-\lambda y,\lambda) \quad \text{as} \quad y\to-\infty.$$

It is easy to prove the analytic properties of $\sigma(\xi,\lambda)$ using the standard ODE theory.

Direct spectral transform

Step 2: We construct the eigenfunction, analytic in the spectral parameter.
For complex λ let us introduce the following complex notations:

$$z = x - \lambda y, \quad \bar{z} = x - \bar{\lambda} y$$

Equation on the wave function takes the form:

$$L\Phi^{\pm}(x, y, \lambda) = 0, \quad L = \partial_y + (\lambda + v_x)\partial_x.$$

and can be written as Beltrami equation:

$$[\partial_{\bar{z}} + b(z, \bar{z}, \lambda)\partial_z]\,\Phi(z, \bar{z}, \lambda) = 0,$$

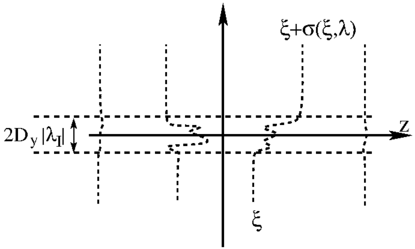
where

$$b(z, \bar{z}, \lambda) = \frac{v_x(z, \bar{z})}{2i\lambda_I + v_x(z, \bar{z})}.$$

Direct spectral transform

It is uniquely solvable without the small norm assumption, and is holomorphic in λ for $\operatorname{Im}\lambda \neq 0$.

What happens if $\operatorname{Im}\lambda \ll 1$, $\operatorname{Im}\lambda < 0$?



Outside a small neighborhood of the real line in the z-plane thef function $\Phi^{-}(x,y,\lambda)$ is holomorphic in z and almost constant on the characteristics. We show that the limit $\hat{\Phi}^{\pm}(z,\lambda) = \Phi^{-}(x,y,\lambda)$ and $\operatorname{Im}\lambda \to -0$ is well-defined and satisfy the shifted Riemann problem:

$$\hat{\Phi}(\xi - i\epsilon, \lambda) \sim \hat{\Phi}(\xi + \tilde{\sigma}(\xi,\lambda) + i\epsilon, \lambda).$$

Thertefore

$$\Phi^-(x,y,\lambda) = \varphi_-(x,y,\lambda) + \chi_-(\varphi_-(x,y,\lambda),\lambda) =$$
$$= \varphi_+(x,y,\lambda) + \chi_+(\varphi_+(x,y,\lambda),\lambda)$$
$$\Phi^+(x,y,\lambda) = \overline{\Phi^-(x,y,\lambda)},$$

and the spectral data $\chi_\pm(\xi,\lambda)$ satisfy the shifted Riemann problem:

$$\sigma(\xi,\lambda) + \chi_+(\xi + \sigma(\xi,\lambda),\lambda) - \chi_-(\xi,\lambda) = 0, \quad \xi \in \mathbb{R},$$
$$\partial_{\bar\xi}\chi = 0 \ \text{ для } \xi \in \mathbb{C}^\pm,$$
$$\chi \to 0 \ \text{ при } |\xi| \to \infty.$$

This procedure is analogous to the construction of non-local Riemann problem data in the classical paper by Manakov dedicated to KP-1.

By analogy with dispersive systems, there are two ways of defining the time dynamics: By introducing the time-dependence in the spectral data:

$$\sigma(\xi,\lambda,t) = \sigma(\xi-\lambda^2 t,\lambda,0),$$
$$\chi_{\pm}(\xi,\lambda,t) = \chi_{\pm}(\xi-\lambda^2 t,\lambda,0),$$

or by introducing the t-dependence in the asymptotic of the wave function. We use the second approach.
The inverse spectral problem equation has the form:

$$\psi_-(x,y,t,\lambda) - H_\lambda \chi_{-I}\Big(\psi_-(x,y,t,\lambda),\lambda\Big) + \chi_{-R}\Big(\psi_-(x,y,t,\lambda),\lambda\Big) = x - \lambda y - \lambda^2 t,$$

where χ_{-R} and χ_{-I} denote the real and imaginary parts of χ_- respectively, H_λ – denotes the Hilbert transform in λ

$$H_\lambda f(\lambda) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda-\lambda'} d\lambda'.$$

In terms of the Hilbert transform analyticity of $\chi_-(\xi,\lambda)$ ξ in the lower half-plane is equivalent to: $\chi_{-\mathrm{R}} - \mathrm{H}_\xi \chi_{-\mathrm{I}} = 0$.

Theorem

Let the spectral data $\chi_-(\xi,\lambda)$ satisfy the following constraints:

1. $\chi_-(\xi,\lambda)$, $\partial_\xi \chi_-(\xi,\lambda)$ are differentiable
2. $$|\partial_\xi \chi_{-\mathrm{R}}(\xi,\lambda)| \leq \frac{1}{4}, \quad |\partial_\xi \chi_{-\mathrm{I}}(\xi,\lambda)| \leq \frac{1}{4}.$$
3. For some $\mathrm{C} > 0$
$$|\chi_-(\xi,\lambda)| \leq \frac{\mathrm{C}}{1+|\lambda|}$$

Then for all $\mathrm{x},\mathrm{y},\mathrm{t} \in \mathbb{R}$, $\mathrm{t} \geq 0$ inverse problem equations are uniquely solvable and $\psi(\mathrm{x},\mathrm{y},\mathrm{t},\lambda) = \mathrm{x} - \lambda \mathrm{y} - \lambda^2 \mathrm{t} + \omega(\mathrm{x},\mathrm{y},\mathrm{t},\lambda)$, where $\omega(\mathrm{x},\mathrm{y},\mathrm{t},\lambda) \in \mathrm{L}^2(d\lambda) \cap \mathrm{L}^\infty(d\lambda)$.

Inverse spectral transform

Theorem

Assume, that we have the following constraints on the inverse spectral data: Let the spectral data $\chi_{-}(\xi,\lambda)$ satisfy the following constraints:

1. $|\partial_{\xi}\chi_{-\mathrm{R}}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right), \quad |\partial_{\xi}\chi_{-\mathrm{I}}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right).$
2. $|\partial_{\xi}^{n}\chi_{-}(\xi,\lambda)| \le \frac{C}{1+|\lambda|^{2+n}}$, $n = 0, 1, 2, 3.$
3. $|\partial_{\xi}^{n}\partial_{\lambda}\chi_{-}(\xi,\lambda)| \le \frac{C}{1+|\lambda|^{3+n}}$, $n = 0, 1.$

Then

- The regularized wave functions $\omega_x, \omega_y, \omega_t \in L^2(d\lambda) \cap L^4(d\lambda)$, $\omega_{xx}, \omega_{xy}, \omega_{xt}, \omega_{yy} \in L^2(d\lambda)$, and $\psi(x, y, t, \lambda)$ satisfy the Lax pair for the Pavlov equation.
- The functions $v_x, v_y, v_{xx}, v_{x,y}, v_{xt}, v_{yy}$ are well-defined and satisfy the Pavlov equation.

Inverse spectral transform: the small norm condition

Let us associate the following constants with the Cauchy data

$$B_0 = \int_{-\infty}^{+\infty} \left[\max_{x \in \mathbb{R}} |v_x(x,y)| \right] dy,$$

$$B_1 = \exp\left[\int_{-\infty}^{+\infty} \left[\max_{x \in \mathbb{R}} |v_{xx}(x,y)| \right] dy \right] - 1,$$

$$B_2 = \left[\int_{-\infty}^{+\infty} \left[\max_{x \in \mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] (1+B_1)^3,$$

$$B_3 = \left[\int_{-\infty}^{+\infty} \left[\max_{x \in \mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] 3(1+B_1)^2 B_2 +$$

$$+ \left[\int_{-\infty}^{+\infty} \left[\max_{x \in \mathbb{R}} |v_{xxxx}(x,y)| \right] dy \right] (1+B_1)^4,$$

Inverse spectral transform: the small norm condition

$$\hat{B}_0 = \left[\int_{-\infty}^{+\infty} \left(\sqrt{\int_{-\infty}^{+\infty} |v_x(x,y)|^2 dx} \right) dy \right] \cdot \frac{1}{\sqrt{1 - B_1}},$$

$$\hat{B}_1 = \left[\int_{-\infty}^{+\infty} \left(\sqrt{\int_{-\infty}^{+\infty} |v_{xx}(x,y)|^2 dx} \right) dy \right] \cdot \frac{1 + B_1}{\sqrt{1 - B_1}}.$$

Inverse spectral transform: the small norm condition

Theorem

Assume that

1. $v(x,y) = 0$ outside the area $-D_x \le x \le D_x$, $-D_y \le x \le D_y$.
2. $B_0 \le \frac{1}{4}$,
3. $B_1 \le \frac{1}{2}$,
4. $8B_0 + 8B_2 + 2\sqrt{2}\hat{B}_0 < \pi$,
5. $2B_1 + \frac{\sqrt{2}}{\pi}(32B_1 + 16\hat{B}_0) + \frac{1}{\pi}(8B_3 + 16B_2^2 + 56B_1 + 16B_1^2)\left(B_0 + \frac{2}{\pi}[2B_0 + \hat{B}_0]\right) < \tan\left(\frac{\pi}{8}\right)$.

Then the unique solubility conditions for the inverse problem are fulfilled.

Inverse spectral transform

By analogy with the standard KP equation the behavior of v_t at $t = 0$ requires an extra investigation.
Another question. What happens, if we start from inverse data without the property:

$$\chi_{-R} - H_\xi \chi_{-I} = 0?$$

It can be shown, that we obtain the same solutions of the Pavlov equation, but the normalization of the wave function will be different from the Jost one at $y \to -\infty$.