Singularity formation on a fluid interface during the Kelvin-Helmholtz instability development

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Introducing remarks

Kelvin-Helmholtz instability is one of the main instabilities in hydrodynamics. This instability is known also as an instability of tangential velocity discontinuity. It is aperiodic one. It also takes place when tangential velocity discontinuity coincides with the fluid interface. The growth rate Γ of this instability is known since Kelvin and Helmholtz (from 19th century). In absence of both gravity g and capillarity α, $\Gamma \sim k$.

When $g, \alpha \neq 0$, Γ has a threshold. For waves exited by wind, the threshold velocity is defined by the minimal phase velocity for the gravity-capillary waves (without wind) and is of the order of 6 m/s. For this wind velocity, the mass appearance of wave crests and forthcoming white caps is observed due to the wave breaking process.

Introducing remarks

In our paper (K. & Lushnikov, 1995) this phenomenon was connected with wave collapse. At small excesses above the threshold, a narrow wave packet is excited in the k-space and resulting equation for envelope is the relativistic Klein-Gordon equation with the negative square of the mass and the " irregular" nonlinearity which leads to collapse.

At $g = \alpha = 0$ the nonlinear stage of this instability is also very interesting. Within the integral Birkhoff-Rott (BR)equation, Moore showed in 1979 that for one fluid the nonlinear stage of the KH instability results in formation of singularities of the root type in a finite time: the surface itself remains smooth, but its curvature becomes singular. These singularities become seeds for the vortex spirals centers.

Introducing remarks

For a fluid interface the motion can be also described by the BR equation together with one additional equation for the vorticity evolution. Numerical solution of these equations showed existence of the Moore-type singularities but theoretical analysis by Baker, Caflisch, and Siegel (1993) contained a number of approximations. The main assumption was connected with ignoring of nonlinear interaction between analytic continuations of solution into the upper and lower half-planes of the corresponding complex variable. This cross term is not small compared to the local one. Here it will be shown that, for the proper choice of variables, cross-terms disappear in a natural way.

Introducing remarks

It is necessary to mention that in two our papers (E.K, M. Spector and V. Zakharov, 1993, 1994) it was shown that this system can be integrated in the small-slope approximation analytically. In this case, from the very beginning the system is pure nonlinear, it does not contain the quadratic part in the Hamiltonian. We showed that the weak nonlinear behavior for any initial (small) data results in the formation of singularities of the root type which are analogous to those found by Moore. These singularities are consistent with the small-slope approximation. In the given paper we will demonstrate that appearance of weak singularities of the root type is the result of the Kelvin-Helmholtz instability development.

Original equations and Hamiltonian description

Let $y = \eta(x,t)$ be the interface position.

For two the velocity potentials $\Phi_{1,2}$ we have two Laplace equations

$$\nabla^2 \Phi_1 = 0, \qquad y < \eta(x,t),$$

$$\nabla^2 \Phi_2 = 0, \qquad y > \eta(x,t)$$

with boundary conditions:

As $y \to \mp\infty$, $\Phi_{1,2} \to V_{1,2}x$, where $V_{1,2}$ are constant velocities.

At the interface $y = \eta(x,t)$:

$$\rho_1 \left(\frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} \right) - \rho_2 \left(\frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} \right) = \frac{\rho_1 V_1^2 - \rho_2 V_2^2}{2},$$

$$\eta_t = \partial_n \Phi_1 \sqrt{1+\eta_x^2} = \partial_n \Phi_2 \sqrt{1+\eta_x^2}.$$

Original equations and Hamiltonian description

It is convenient to introduce the auxiliary velocity potentials,

$$\tilde{\Phi}_{1,2} = \Phi_{1,2} - V_{1,2}x,$$

These equations are Hamiltonian ones:

$$\psi_t = -\frac{\delta H}{\delta \eta}, \qquad \eta_t = \frac{\delta H}{\delta \psi},$$

where $\psi(x,t) \equiv \rho_1\psi_1 - \rho_2\psi_2$, $\psi_{1,2} = \tilde{\Phi}_{1,2}|_{y=\eta}$ and the Hamiltonian coincides with the total energy of the system,

$$H = \rho_1 \int_{y\le\eta} \frac{(\nabla\Phi_1)^2 - V_1^2}{2} dx\,dy + \rho_2 \int_{y\ge\eta} \frac{(\nabla\Phi_2)^2 - V_2^2}{2} dx\,dy.$$

These variables, $\psi(x,t)$ and $\eta(x,t)$, generalize the canonical variables introduced by V.E. Zakharov for the surface waves.

Weakly nonlinear regime

First it is convenient to consider the system dynamics in the center-of-mass frame

$$\rho_1 V_1 + \rho_2 V_2 = 0.$$

Then expand H in series relative to canonical variables assuming $|\eta_x| \ll 1$:

$$H = H_0 + H_{int}.$$

Here

$$H_0 = \frac{1}{2(\rho_1 + \rho_2)} \int \psi \hat{k} \psi \, dx - \frac{c^2(\rho_1 + \rho_2)}{2} \int \eta \hat{k} \eta \, dx$$

where $\hat{k} = \partial_x \hat{H}$, $\hat{H}$ is the Hilbert transform operator.

Weakly nonlinear regime

Hence we arrive at the (linear) Kelvin-Helmholtz instability:

$$\omega^2 = -c^2 k^2 < 0$$

where $c = V_1\sqrt{\rho_1/\rho_2} = -V_2\sqrt{\rho_2/\rho_1}$. This instability is aperiodic.

In dimensionless variables $H = H_0 + H_3$ can be written as

$$H = \frac{1}{2}\int \left[\psi \hat{k}\psi - \eta \hat{k}\eta\right] dx + \frac{A}{2}\int \eta \left[(\psi_x)^2 - (\hat{k}\psi)^2 + (\eta_x)^2 - (\hat{k}\eta)^2\right] dx$$

$$-\sqrt{1-A^2}\int \eta \left[\eta_x \hat{k}\psi + \psi_x \hat{k}\eta\right] dx.$$

Here

$$A = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}$$

is the Atwood number.

Weakly nonlinear regime

Let us perform a canonical transformation from the variables ψ and η to new ones

$$f = (\psi + \eta)/2, \qquad g = (\psi - \eta)/2,$$

so that

$$f_t = \frac{\delta H}{\delta g}, \qquad g_t = -\frac{\delta H}{\delta f},$$

and $H \to 2H$. In terms of the new variables,

$$\begin{aligned} H \;=\; & \int f\hat{k}g\,dx + (A/2)\int (f-g)\left[(f_x)^2 - (\hat{k}f)^2 + (g_x)^2 - (\hat{k}g)^2\right]dx \\ & -\sqrt{1-A^2}\int (f-g)\left[f_x\hat{k}f - g_x\hat{k}g\right]dx. \end{aligned}$$

Weakly nonlinear regime

In linear regime we have two separated equations,

$$f_t - \hat{k} f = 0, \qquad g_t + \hat{k} g = 0.$$

f grows exponentially, but g describes damping. Hence, at times of order of the inverse growth rate, $g \ll f$. Then

$$\begin{aligned} H &= \int f \hat{k} g \, dx + (A/2) \int (f - g) \left[(f_x)^2 - (\hat{k} f)^2 \right] dx \\ &\quad - \sqrt{1 - A^2} \int (f - g) \left[f_x \hat{k} f \right] dx. \end{aligned}$$

Weakly nonlinear regime

As the result, the corresponding equations of motion have the form

$$f_t - \hat{k}f = (A/2)\left[(\hat{k}f)^2 - (f_x)^2\right] + \sqrt{1-A^2}\left[f_x\hat{k}f\right],$$

$$\begin{aligned} g_t + \hat{k}g &= (A/2)\left[(\hat{k}f)^2 - (f_x)^2 + 2(ff_x)_x + 2\hat{k}(f\hat{k}f)\right] + \sqrt{1-A^2}\left[\hat{k}(ff_x) - \right. \\ &\quad -A\left[(gf_x)_x + \hat{k}(g\hat{k}f)\right] - \sqrt{1-A^2}\left[\hat{k}(gf_x) - (g\hat{k}f)_x\right]. \end{aligned}$$

Thus, Eq. for f is autonomous, while the equation is linear with respect to g.

Root singularities

Expand f by two analytic continuations into the upper and lower half-planes of x, $f_\pm$:

$$f = f_+ + f_-,$$

where $f_\pm = \hat{P}_\pm f$, and $\hat{P}_\pm = (1 \mp i\hat{H})/2$ are corresponding projectors . Due to the properties,$\hat{P}_\pm^2 = \hat{P}_\pm$ and $\hat{P}_\pm \hat{P}_\mp = 0$, the nonlinear terms split into a sum of functions analytically continuable into the upper/lower half-planes:

$$\begin{aligned}(\hat{k}f)^2 - (f_x)^2 &\equiv (\hat{H}f_x)^2 - (f_x)^2 = -2\left(f_{+x}^2 + f_{-x}^2\right), \\ f_x\hat{k}f &\equiv -f_x\hat{H}f_x = -i\left(f_{+x}^2 - f_{-x}^2\right).\end{aligned}$$

Hence the equations for $f_\pm$ are separated into two independent equations.

Root singularities

In particular, for $f_+ \equiv F$, we have an autonomous equation:

$$F_t + iF_x = -e^{i\gamma} F_x^2,$$

where $\gamma = \text{arccos} A$. Differentiating this equation with respect to x leads to the equation of the Hopf-type:

$$V_t + iV_x = -2e^{i\gamma} V V_x,$$

where $V = F_x$ has a meaning of the complex velocity. The solution is written in the implicit form:

$$V = V_0(\tilde{x}), \qquad x = \tilde{x} + it + 2e^{i\gamma} V_0(\tilde{x}) t,$$

where $V_0(x) = V|_{t=0}$, and $\tilde{x}$ is the Lagrangian coordinate. V has all singularities in the lower half-plane.

Root singularities

As was shown in our papers (E.K., M. Spector and V. Zakharov) every point singularity transforms at $t > 0$ into a cut with the movable branch points. Their locations are defined by the condition $\partial x / \partial \tilde{x} = 0$, i.e.,

$$1 + 2e^{i\gamma} V_0'(\tilde{x}) t = 0.$$

When the most rapid branch point reaches the real axis, the analyticity of V breaks down and a singularity appears in the solution. The motion of the branch point $\tilde{x} = \tilde{X}(t)$ in the x plane is described by

$$x = X(t) = \tilde{X}(t) + it + 2e^{i\gamma} V_0\left(\tilde{X}(t)\right) t.$$

Root singularities

The collapse time t_c is defined from $\text{Im}\, X(t_c) = 0$.
The expansion near singular point gives in the leading order

$$V = V_0(\tilde{x}_c) + V_0'(\tilde{x}_c)\delta\tilde{x} + \ldots,$$

$$\delta x = i\delta t + 2e^{i\gamma}V_0(\tilde{x}_c)\delta t + t_c e^{i\gamma}V_0''(\tilde{x}_c)(\delta\tilde{x})^2 + \ldots,$$

where $\delta t = t - t_c$, $\delta x = x - x_c$, and $\delta\tilde{x} = \tilde{x} - \tilde{x}_c$.
Excluding the parameter $\delta\tilde{x}$ yields

$$V(x,t) = V_0(\tilde{x}_c) + V_0'(\tilde{x}_c)\left[\frac{\delta x - \left(i + 2e^{i\gamma}V_0(\tilde{x}_c)\right)\delta t}{t_c e^{i\gamma}V_0''(\tilde{x}_c)}\right]^{1/2} + \ldots$$

Root singularities

Hence one can see that the derivatives V_x and V_t become singular. As a result, the boundary shape acquires root singularities. Hence it follows

$$V_x(x,t) \approx \frac{V_0'(\tilde{x}_c)}{2\sqrt{t_c e^{i\gamma} V_0''(\tilde{x}_c)}} \left[\delta x - \left(i + 2e^{i\gamma} V_0(\tilde{x}_c)\right)\delta t\right]^{-1/2},$$

i.e., in the general case, $V_x(x_c,t) \sim |\delta t|^{-1/2}$.

For the interfacial curvature, specified as $\eta_{xx}(1+\eta_x^2)^{-3/2}$, is of

$$\eta_{xx} \approx 2\,\mathrm{Re}\,F_{xx} = 2\,\mathrm{Re}\,V_x.$$

Root singularities

Hence one can find the following universal relationship for the curvature in the singular point vicinity:

$$\eta_{xx} \approx \text{Re}\left\{ \frac{V_0'(\tilde{x}_c)}{\sqrt{t_c e^{i\gamma} V_0''(\tilde{x}_c)\,(\delta x - i\delta t)}} \right\}.$$

REMARK 1

In the case when both velocities $V_{1,2} = 0$ equation of motion for analytical continuation of $u \sim \psi_x^+$ has the form of the Hopf equation

$$u_t + uu_x = 0.$$

Hence it is easily to see that at $\operatorname{Im} x = 0$ this equation for real and imaginary parts, $u = V + in$, is transformed into two equations

$$n_t + (nV)_x = 0, \;\; V_t + VV_x = \frac{1}{2}(n^2)_x.$$

This system (of the hydrodynamic type) represents quasiclassical limit of the quintic NLS equation which is the critical model for the collapse formation.

References

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