Bounded solutions of integrable wave equation via Dressing Method

V. E. Zakharov, D. V. Zakharov

1 „Traditional" Riemann–Hilbert problem on the real axis" $-\infty < k < \infty$

$\chi(k)$ analytic on the k–plane with exception of real axis

$$\chi(k) \to 1 + \frac{\chi_0}{k} + \dots \quad \text{at} \quad k \to \infty$$

$$\chi^{\pm}(k) = \lim_{\varepsilon \to 0} \chi(k \pm i\varepsilon) \qquad k \text{ is real}$$

Function $\chi(k)$ obeys the „traditional" Riemann–Hilbert probem if for all real k

$$\chi^{+}(k) - \chi^{-}(k) = \frac{1}{2} C(k)\, e^{2ikx} \left(\chi^{+}(-k) + \chi^{-}(-k) \right)$$

x is a parameter

Function $f(k)$ has a spectral representation

$$f(k,x) = 1 + \frac{1}{2\pi i} \int \frac{R(x,q)}{q-k}\, dq \qquad \operatorname{Im} k \neq 0$$

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_k e^{ikx}\, dk$$

let us denote $K(x,y) = R(x, y-x)$

$$K(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(x,q)\, e^{iq(y-x)}\, dq$$

Marchenko equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,z)\, F(z+y)\, dz = 0$$

$$\underline{u(x)} = -2 \frac{d}{dx} K(x,x) = -2 \frac{d}{dx} y_0(x)$$

Equivalent $\bar{\partial}$ – problem

$$\frac{1}{2i} \frac{\partial f}{\partial \bar{k}} = C(k,\bar{k})\, e^{2ik} f(-k,-\bar{k}) \qquad k = k_R + i k_I$$

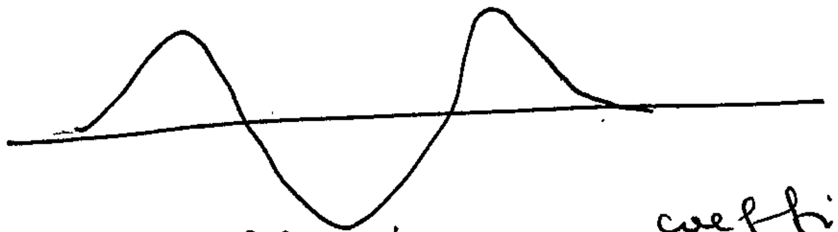
$$C(k,\bar{k}) = C(k_R)\, \delta(k_I)$$

$$\varphi(x,k) = \chi(x,k)\, e^{-ikx}$$

$$\varphi_{xx} + (k^2 - u(x))\varphi = 0$$

$u(x)$

$D(k)\, e^{-ikx} \leftarrow$ $\qquad\qquad \leftarrow e^{-ikx} + C(k) e^{ikx}$



$C(k)$ — reflection coefficient $\qquad |C_k|^2 < 1$

$C_{-k} = C_k^*$

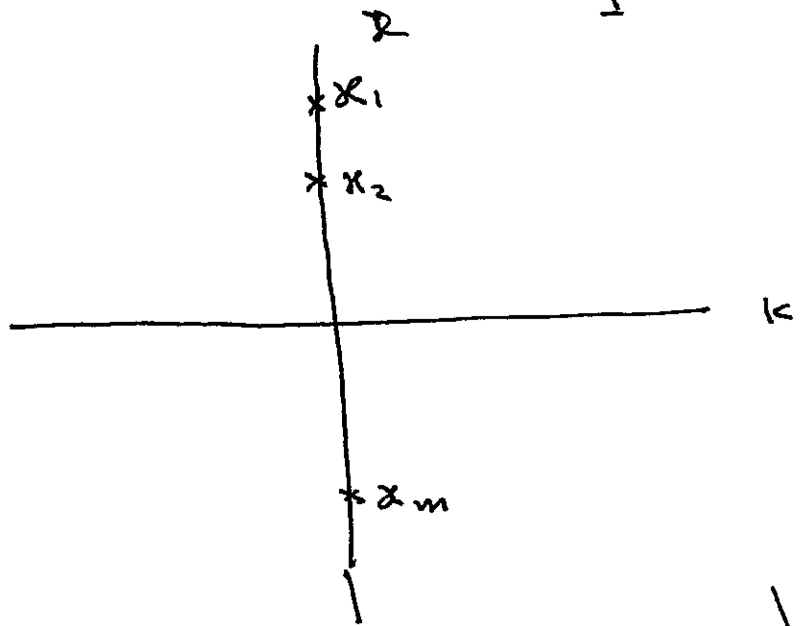
Let $C(k) \equiv 1$

$$C(k,\bar{k}) = \sum_{n=1}^{N} \rho_n(x)\, \delta(k - i\varkappa_n)$$

$$\chi = 1 + \frac{1}{2\pi} \sum \frac{\rho_n(x)}{k - i\varkappa_n}$$

$$\rho_n + C_n e^{-2\varkappa_n x}\left(1 + \frac{1}{2\pi} \sum_{n=1}^{N} \frac{\rho_m}{\varkappa_n + \varkappa_m}\right)$$

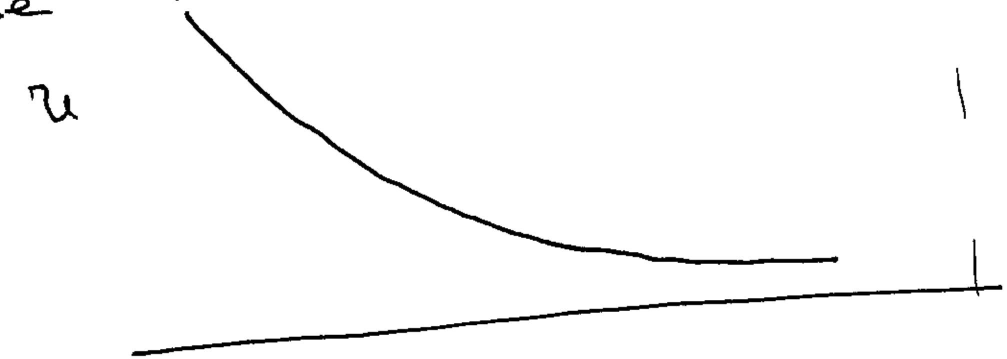
N — solitonic solution



$\varkappa_n + \varkappa_m \neq 0$

One can assume $|c_k| = 1 = e^{i\varphi_k}$

reflectionless potential



$c_k = 0$

One can assume

$$f = \frac{1}{2\pi} \int_0^\infty \frac{f(x,\varkappa)}{k - i\varkappa}\, d\varkappa + 1$$

A combination is possible, but $u \to 0$ at $x \to +\infty$

Sectret of Butterfly

$|\varsigma_k| = 1 \qquad k_2 < k < k_1 \;, \quad -k_1 < k < -k_2$

$|\varsigma_k| = 0$ outside of this domain

What is $u(x)$?

2. Symmetric Riemann- Hilbert problem

(Thereaf ter $x \to k$)

$$\chi(k) = 1 + \frac{A(x)}{k} + \frac{1}{2\pi i}\int_{-L}^{L} \frac{\rho(q,x)}{q-k}\,dq$$

$$\chi(\cancel{k}) \to 1 + \frac{\chi_0(x)}{k}$$

$$\chi_0(x) = A(x) - \frac{1}{2\pi i}\int_{-L}^{L} \rho(q,x)\,dq$$

$$\chi^{\pm}(k) = 1 + \frac{1}{2\pi i}\fint_{-L}^{L} \frac{\rho(q,x)}{q-k}\,dq \pm \frac{1}{2}\rho(k,x) = \lim_{k = k \pm i\varepsilon} \chi(k)$$

$$\chi^{\pm}(-k) = 1 + \frac{1}{2\pi i}\fint_{-L}^{L} \frac{\rho(q,x)}{q+k}\,dq \pm \frac{1}{2}\rho(-k,x)$$

$$y^{\pm}(-k) = \lim_{\varepsilon \to 0} y^{\pm}(-k^{*}) \qquad k \Rightarrow k \pm i\varepsilon$$

The symmetric Riemann–Hilbert problem is defined as follows

$$e^{-kx - d(k)}\, y^{\pm}(k) = e^{kx + d(k)}\, y^{\pm}(-k)$$

$$e^{-kx - d(k)}\, \rho(k) = \rho(-k)\, e^{kx + d(k)}$$

$$d(-k) = -d(k)$$

$$\rho = \underset{k>0}{\rho^{+}(k)} + \underset{k<0}{\rho^{-}(k)}$$

$$\rho^{+}(k) = i f(k)\, e^{kx + d(k)}$$

$$\rho^{-}(-k) = i f(k)\, e^{-kx - d(k)} \qquad \Omega \text{ — spectral set}$$

$$f(k) \neq 0 \qquad k \in \Omega$$

$$f(k) = 0 \qquad k \bar{\in} \Omega$$

f satisfies to equation

$$f'' - 2k f' - u(x) f$$

$\varphi = f e^{kx}$ is a solution of the Schrödinger

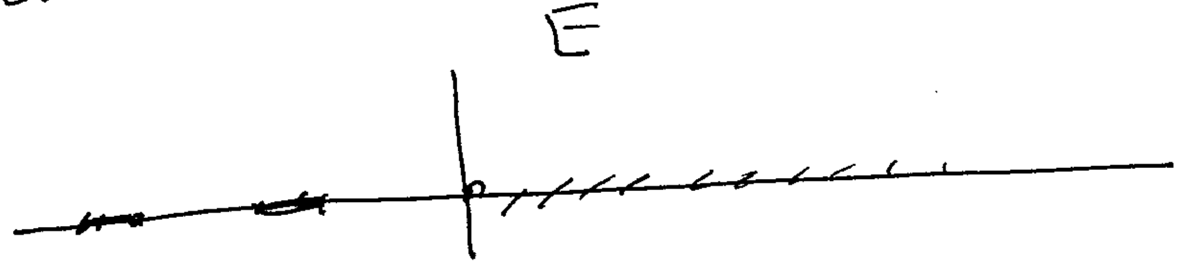
equation

$$\frac{d^2\varphi}{dx^2} - (k^2 + u(x))\varphi = 0$$

$E = -k^2$ $\qquad$ $k \in \Omega$

$k^2 \in$ – negative part of

the spectrum



$$\alpha(k) = \alpha_0(k) - 4k^3 t \qquad \varphi_t = (2u - 4k^2)\varphi_x - u_x \varphi$$

$$u_t - 6 u u_x + u_{xxx} = 0$$

$$\alpha(k) = \alpha_0(k) - 4k^3 t + \sum \alpha_n k^{2n+1} t_n$$

u ~~bet~~ obeys the KdV hierarchy

3. Integral equation

Let $k \in \Omega$ if $k_2 < k < k_1$ $\quad -k_1 < k < -k_2$

$k = k_0 + p$ $\quad g(k_0+p) \to g(p)$ $\quad -\Delta < p < \Delta$

$$\frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{g(q,x)}{q-p} \cosh\left[(p-q)x - \alpha(q) + \alpha(p)\right] dq -$$

$$- \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \frac{g(q,x)}{2k_0+p+q} \cosh\left[(p+q+2k_0)x + \alpha(p) + \alpha(q)\right] dq +$$

$$+ \frac{A(x)}{k_0+p} \cosh\left[(k_0+p)x + \alpha(p)\right] = \sinh\left[(k_0+p)x + \alpha(p)\right]$$

This is a Fredholm equation of the first type

$$\hat{H}_0 g = \hat{H}_1 g + F(p, x) = h(p, x)$$

$$H_0 = \frac{1}{\pi} \int_{-\Delta}^{\Delta} \frac{1}{q-p} g(p)\, dp$$

$$\hat{H}_1 g = \int_{-\Delta}^{\Delta} H_1(p, q, x)\, g(q, x)\, dq$$

$$\hat{H}_1 = \frac{1}{\pi} \left[\frac{\cosh\left[(2k_0 + p + q)x + \alpha(p) + \alpha(q)\right]}{2k_0 + p + q} + \right.$$

$$\left. + \frac{1 - \cosh\left((p-q)x + \alpha(p) - \alpha(q)\right)}{p - q} \right]$$

$\hat{H}_1$ — completely continuous operator

$$F(p, x) = -\frac{A(x)}{k_0 + p} \cosh\left[(k_0 + p)x + \alpha(p)\right] + $$

$$+ \sinh\left[(k_0 + p)x + \alpha(p)\right]$$

Thereafter $\Delta = 1$

$$\frac{1}{\pi} \int_{-1}^{1} \frac{1}{q-p} \frac{dq}{\sqrt{1-q^2}} = 0$$

Hence
$$\int_{-1}^{1} \frac{h(p,x)}{\sqrt{1-p^2}} dp = 0$$

$$A(x) = \frac{\displaystyle\int_{-1}^{1} \frac{H_1(p,q)\, g(q,x)\, dq\, dx}{\sqrt{1-p^2}} + \int_{-1}^{1} \frac{\sinh\left[(k_0+p)x + d(p)\right]}{\sqrt{1-p^2}} dp}{\displaystyle\int_{-1}^{1} \frac{\cosh\left[\left(k_0+p\right)x + d(p)\right)}{(k_0+p)\sqrt{1-p^2}} dp}$$

Additional condition $g(\pm\Delta) = 0$

$$g(p) = \frac{1}{\sqrt{1-p^2}} \sum_{n=1}^{\infty} g_n T_n(p))$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{T_n(s)}{\sqrt{1-s^2}\,(s-x)}\,ds = -U_{n-1}(x)$$

$$T_n = \cos(n \arccos x)$$

$$U_n = -\frac{\sin(n \arccos x)}{\sin x}$$

$$\hat{H}g = -\sum g_n U_{n-1}$$

$$g_n = -2\int_{-1}^{1}\sqrt{1-q^2}\;U_{n-1}(q)\,h(q,x)\,dq$$

This is an infinite system of linear algebraic equation. For moderate x it can be solved numerically

4. Let $P(x)$ is the Weierstrass function with periods $\omega_1 = 2\omega$ $\omega_2 = 2i\omega'$

The Lame equation

$$\left(\frac{d^2}{dx^2} - 2p(x)\right)\varphi = P(a)\varphi$$

has a solution

$$\varphi(x,a) = \frac{\sigma(x - i\omega' + a)\,\sigma(-i\omega')}{\sigma(x - i\omega')\,\sigma(a - i\omega')}\, e^{-\zeta(a)x}$$

Properties of φ

① $\varphi(x,a)$ is double-periodic

$$\varphi(x, a+2\omega) = \varphi(x,a) \qquad \varphi(x, a+2i\omega') = \varphi(x,a)$$

② $\varphi(x,a) \simeq \dfrac{A(x)}{a - i\omega'} + \dots$

$A(x) = A^*(x) = -A(x)$ — real and odd

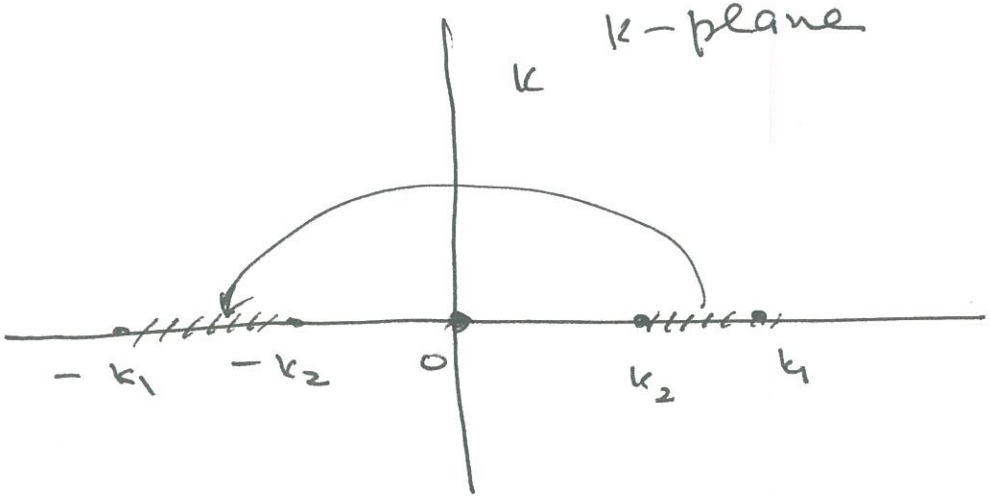
The map ~~to~~ k-plane

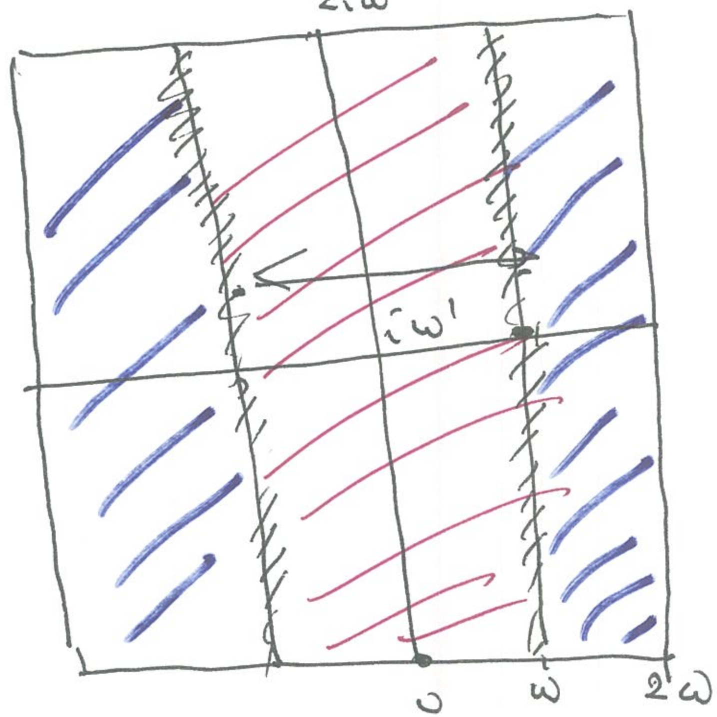
$$k^2 = P(a) - e_3 \qquad P_x^2 = 4(p-e_1)(p-e_2)(p-e_3)$$

$$k'^2 = (k^2 - k_1^2)(k^2 - k_2^2)$$

$$k_1^2 = e_1 - e_3 \qquad k_2^2 = e_2 - e_3 \qquad k_1^2 > k_2^2 > 0$$

k-plane





$$K = \frac{1}{\operatorname{sn} a}$$

$$\Delta = \frac{k_2 - k_1}{2} \qquad k_0 = \frac{k_1 + k_2}{2}$$

Two-sheet Riemann surface

$$\psi = \varphi e^{kx} \to 1 \quad \text{at} \quad |k| \to \infty$$

$$\psi(k, x) = 1 + \int_{-\Delta}^{\Delta} g(q, x) \left[\frac{e^{(q+k_0)x}}{q-k} + \frac{e^{-(q+k_0)x}}{q+k_0+k} \right] dq$$

$$\boxed{\alpha = 0 \ !}$$

$$\alpha = kx_0 \qquad x \to x + x_0$$

$$g(-\Delta) = g(\Delta) = 0$$

$\varphi(x, a)$ are real at

$\operatorname{Im} a = 0 \qquad a = i\omega' + p \qquad p$ — is real

On the vertical straight line $\operatorname{Re} a = \omega$

$$a = i\omega' + \omega + p \qquad \varphi(x - p) = \overline{\varphi}(x, p)$$

$$i g = \varphi(x, p) - \varphi(x - p) \qquad g \text{ is real}$$

$\chi(k, x)$ satisfies to the symmetric Riemann-Hilbert problem

in the limit $\omega \to \infty$ $k_2 \to k_1$

$g \to 0$

$$A = k_0 \tanh k_0 x$$

$$u \to -\frac{2k_0^2}{\cosh^2 k_0 x} \quad \text{— soliton!}$$

We generalized the Dressing method to one-zone potential. In the n-zone case Ω consists of a finite number of intervales, while $\alpha(k)$ — piecewise linear function.

The most interesting problem $\alpha(k)$ is a random function $\underline{k_1} < k < k_2$.

The theory of solitonic gas —

integrable turbulence