Correlation functions

of the anisotropic Kagome Ising antiferromagnet

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- model lattice antiferromagnet
- procedure transfer matrix, Graßmann variables
- technicalities $q \in [0, \frac{\pi}{4}] \;\rightarrow$ matrix structure

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This talk is dedicated to the memory of Yuri A. Bychkov 1934-2012

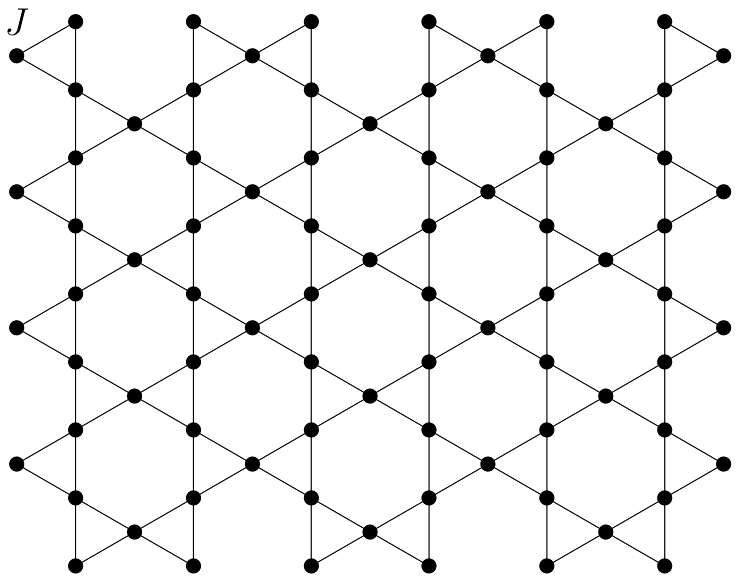
model: antiferromagnet

$$\mathcal{H} = \sum_{<\mathbf{x},\mathbf{x}'>} J_{\mathbf{x},\mathbf{x}'}\, \mathbf{s}_{\mathbf{x}}\, \mathbf{s}_{\mathbf{x}'}$$

□–Heisenberg: spontaneous magnetization

△–Heisenberg: spontaneous magnetization

Kagome Heisenberg: ?? num. evidence: no LRO



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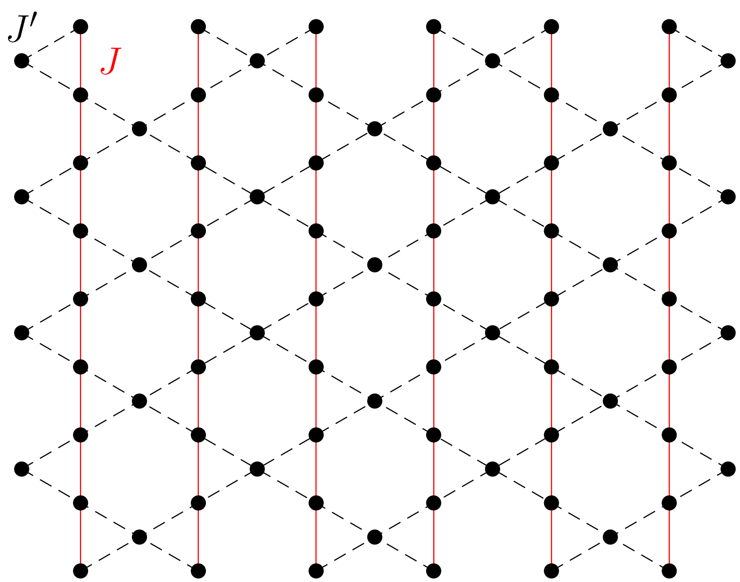
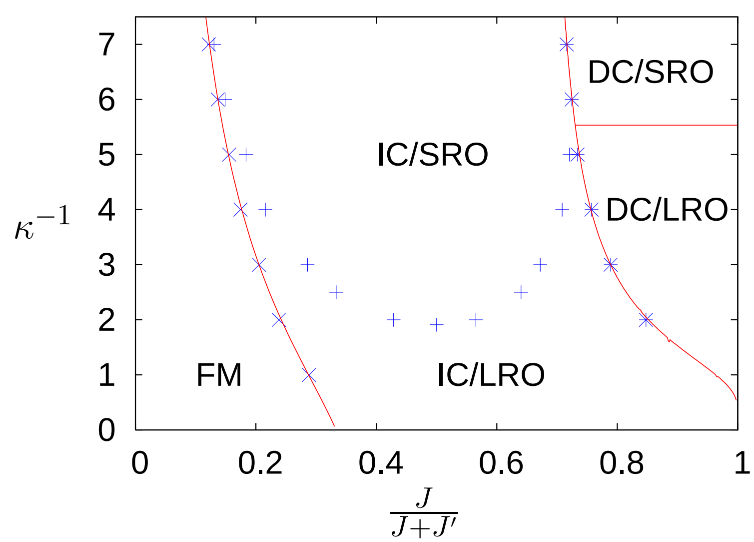


anisotropy: chain spins / middle spins

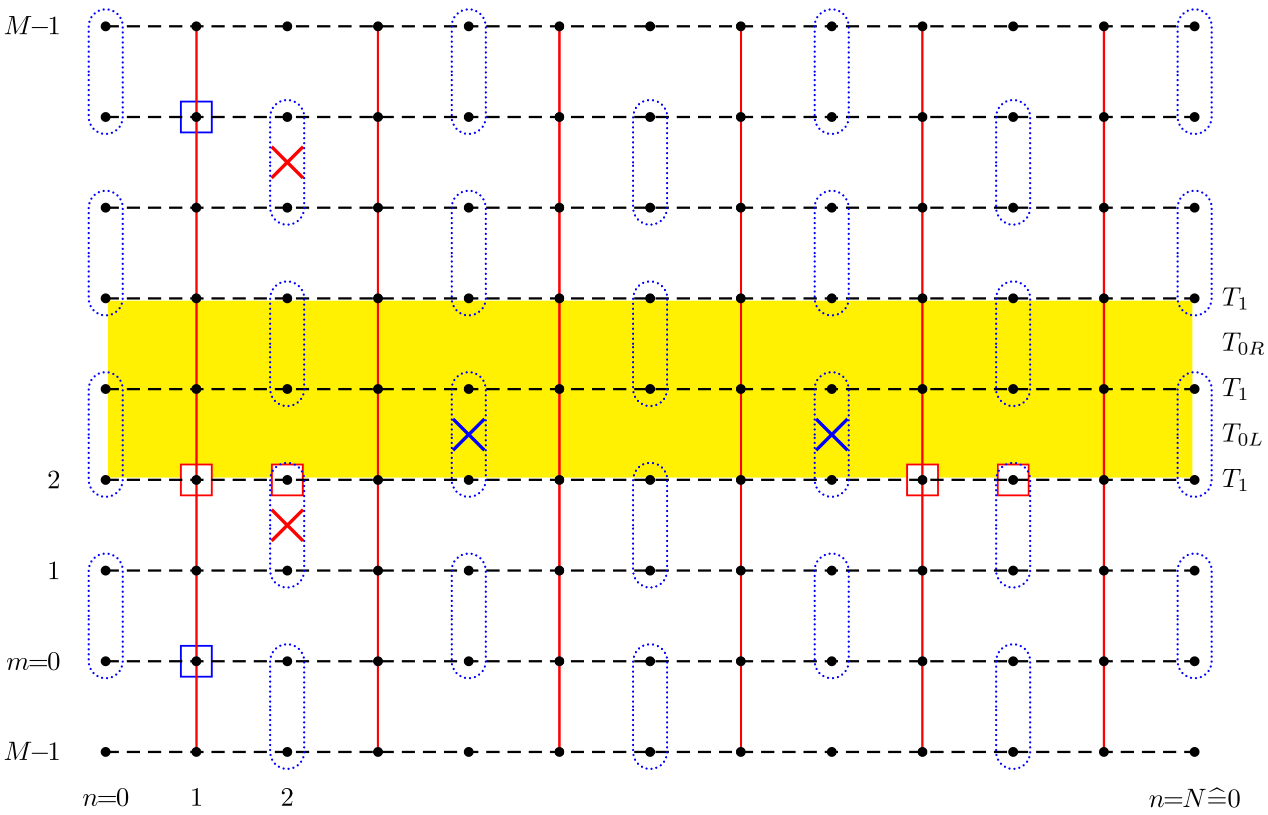
T Yavors'kii, WA, HUE: PRB (2007)



realization: Volborthite [Yoshida et al. Nat Comm **3**, 860 (2012)]

model: Ising antiferromagnet

splitting of middle spins: Kano & Naya (1953)



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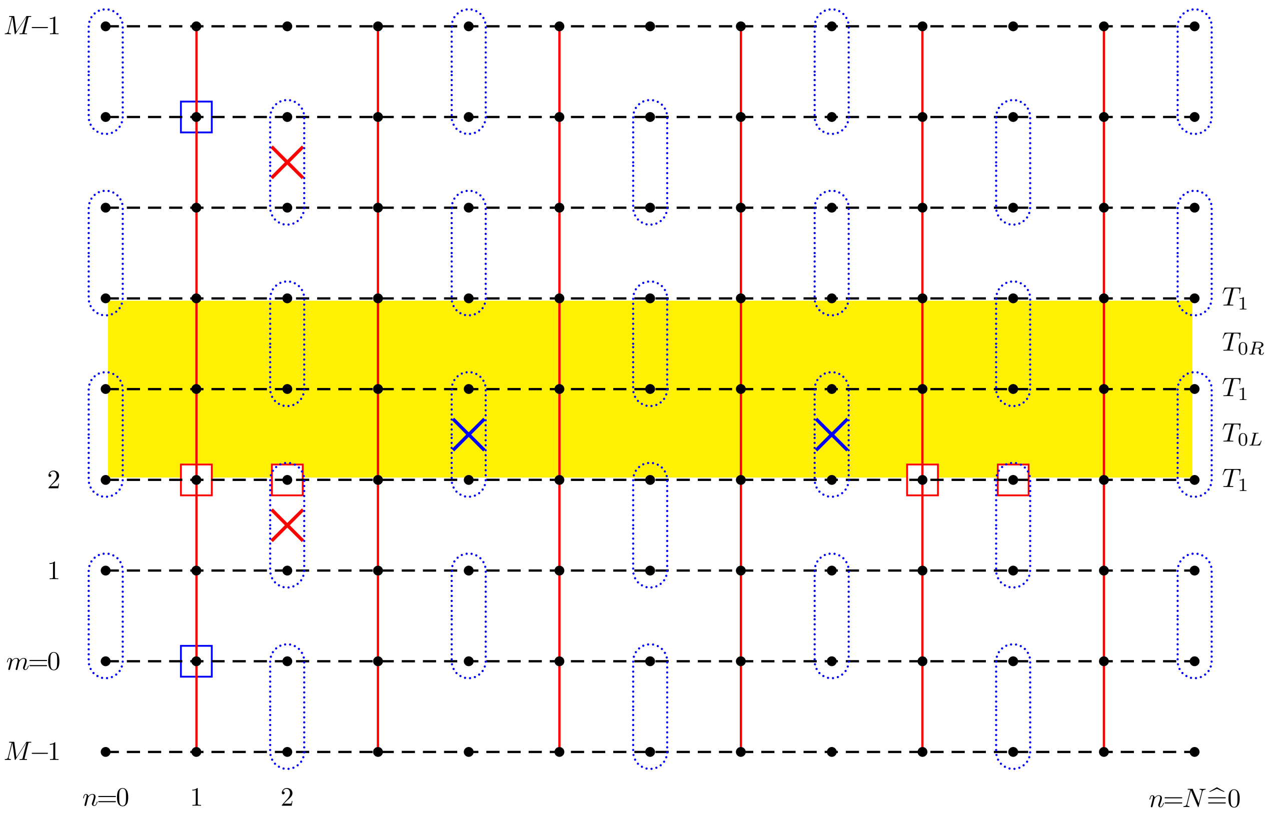

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$\Phi := \tanh \beta J' \quad \in [0,1]$

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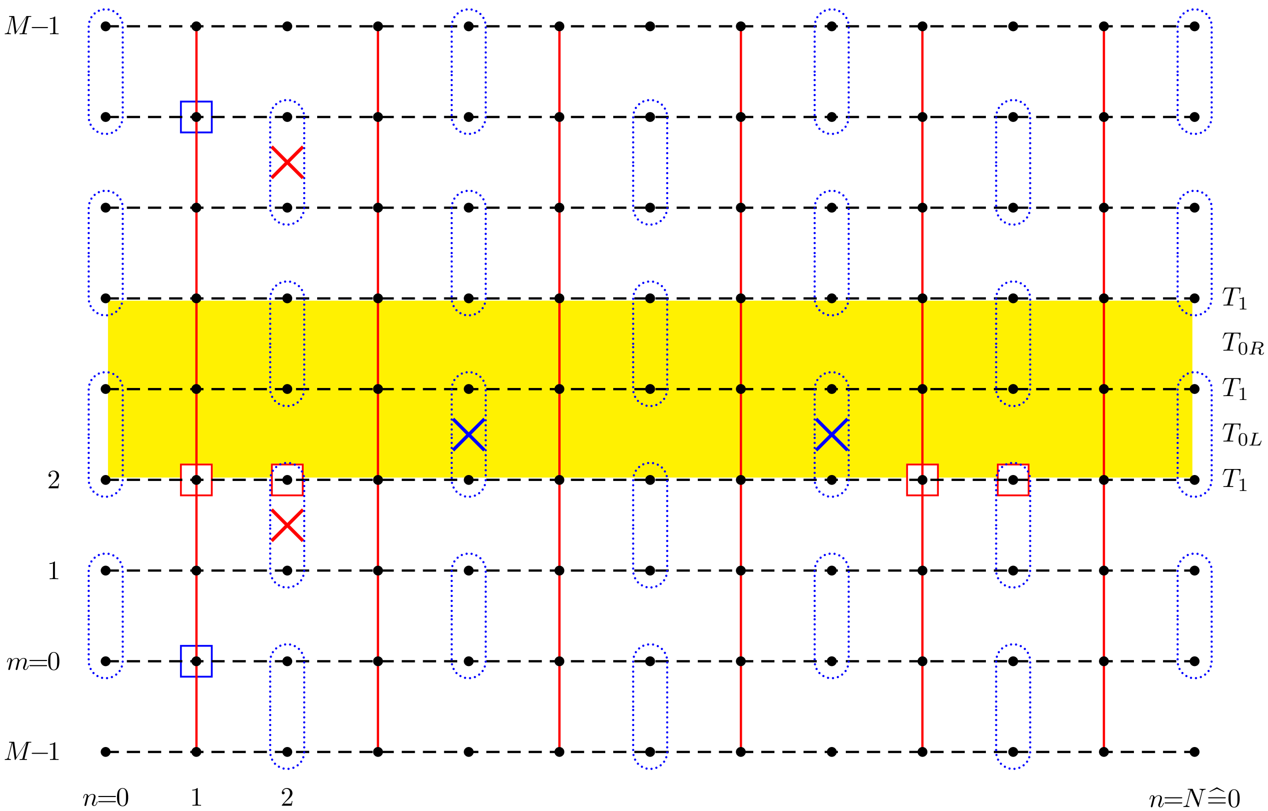

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transfer matrix $\Leftrightarrow$

$$\chi_{\color{red}\times}(x) = \langle 0_{\Leftrightarrow}|\, \mathcal{C}_0 \mathcal{C}_x \,|0_{\Leftrightarrow}\rangle$$

$$\chi_{\color{blue}\times}(x) = \sum_{\alpha} \langle 0_{\Leftrightarrow}|\, \mathcal{C}_0 \,|\alpha\rangle\langle\alpha|\, \mathcal{C}_x \,|0_{\Leftrightarrow}\rangle \; e^{-(\epsilon_\alpha - \epsilon_0^{\Leftrightarrow})\,x}$$

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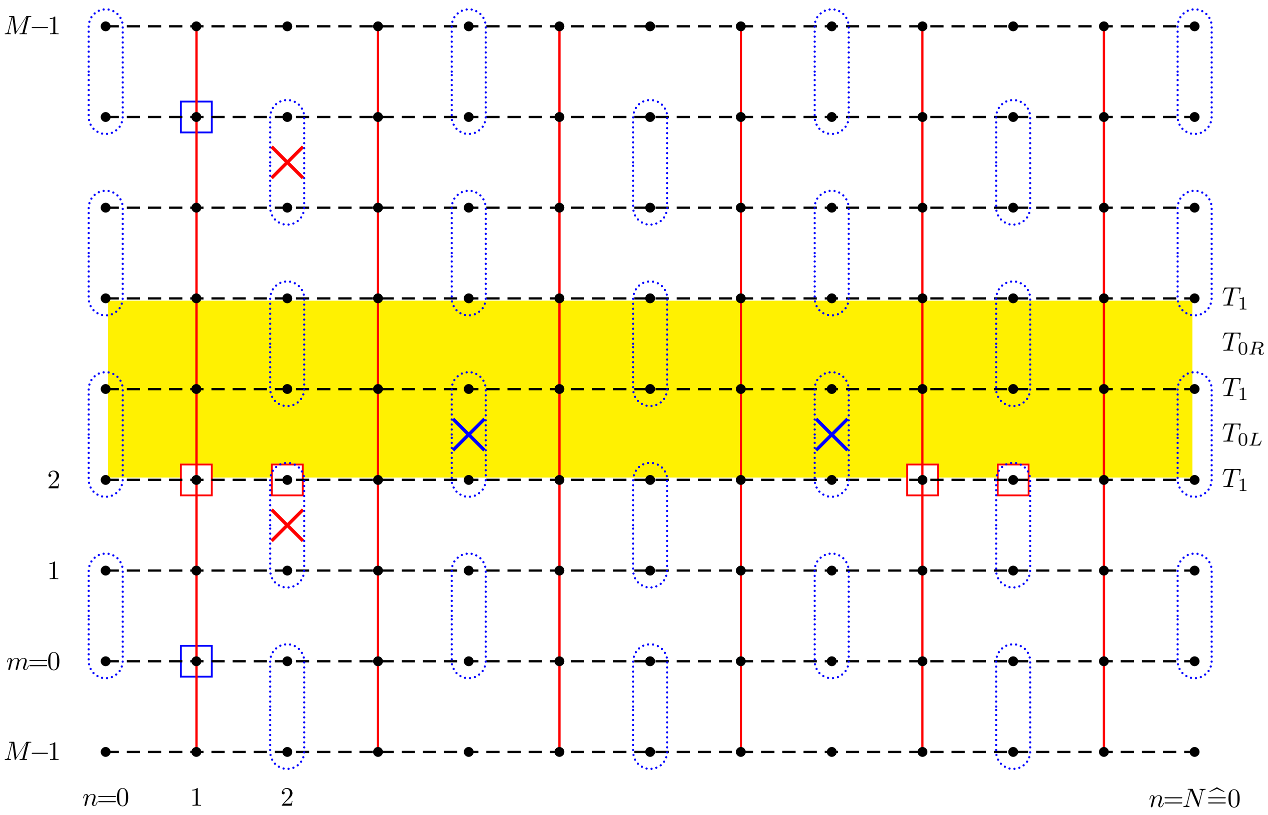

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transfer matrix $\Updownarrow$

$\chi_{\square}(x) = \langle 0_{\Updownarrow} | \, \mathcal{C}_0 \mathcal{C}_x \, | 0_{\Updownarrow} \rangle$

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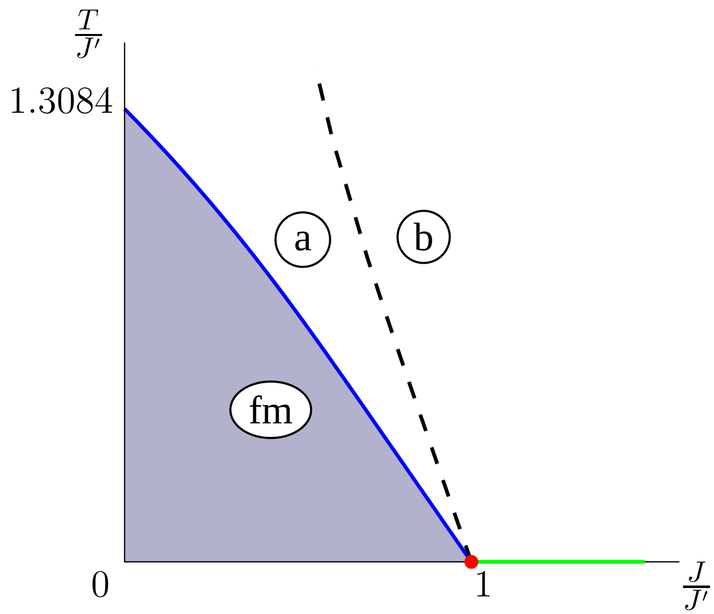
model: phasediagram



transfer matrix ⇔ see *J. Stat. Mech.* (2011) P09002

results for $\chi_{\times}$:

fm ordered region

a $\chi_{\times} \sim \frac{1}{\sqrt{n}}\, e^{-n/\xi_1}$

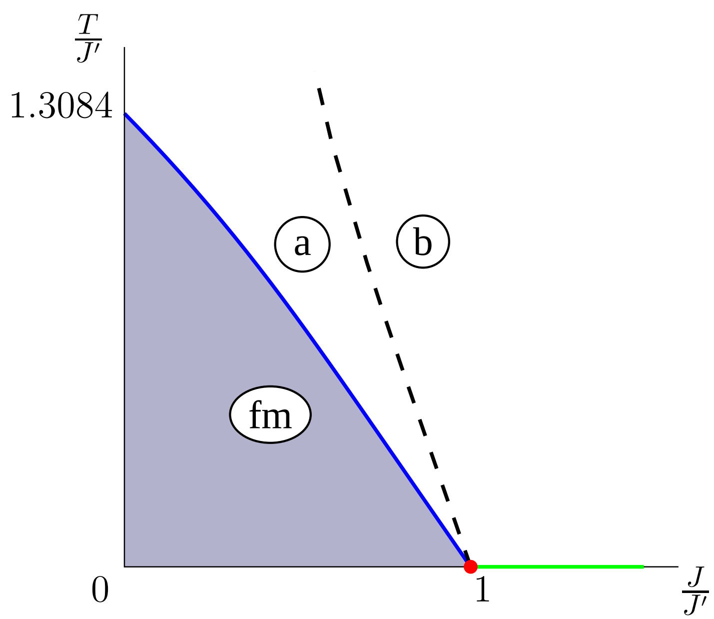
disorderline $\chi_{\times} = 0$

b $\chi_{\times} \sim -\frac{1}{\sqrt{n}^3}\, e^{-n/\xi_2}$

● $\ldots$

— $\ldots$

model: phasediagram



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now: transfer matrix ⇕

energy identical to that of the previous approach (⇔) ✔ (s. Kano & Naya (1953))

3 new correlation functions: $\chi_{\square}(i,j)$ odd-odd, odd-even and even-even

model: status of the calculation of $\chi_{\square}$

$\chi_{\square}(ij) = \sqrt{\det(\mathbf{g})}$ **g** is generalized Toeplitz matrix: $\mathbf{g} = \begin{pmatrix} \mathbf{g}^{\downarrow\downarrow} & \mathbf{g}^{\downarrow\uparrow} \\ \mathbf{g}^{\uparrow\downarrow} & \mathbf{g}^{\uparrow\uparrow} \end{pmatrix}$ and $\mathbf{g}^{\downarrow\downarrow} = \begin{pmatrix} g^{\downarrow\downarrow}_{jj} & \cdots & g^{\downarrow\downarrow}_{j\,i-1} \\ \vdots & & \vdots \\ g^{\downarrow\downarrow}_{i-1\,j} & \cdots & g^{\downarrow\downarrow}_{i-1\,i-1} \end{pmatrix}$ etc.

g is a <u>finite</u> dimensional matrix of size $2(i-j) \times 2(i-j)$ with the matrixelements

$$g^{\downarrow\uparrow}_{n\,n'} = -\delta_{n\,n'} + \int_{-\pi}^{\pi} \frac{dq}{2\pi}\, e^{-iq(n-n')} \left(2\,A_d(q) + \left[(-)^n + (-)^{n'}\right] A_a(q) + \left[(-)^n - (-)^{n'}\right]\, \mathcal{S}(q)\, A_a(q)\right)$$

etc.

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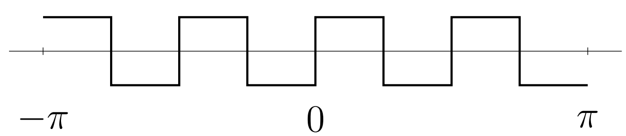
and $$A_d(q) = \frac{v_M \det \begin{pmatrix} v_M + iM_{ez} & -iM_{es} & -iM_{ea} \\ iM_{fz} & v_M - iM_{fs} & -iM_{fa} \\ iM_{siz} & -iM_{sis} & v_M - iM_{sia} \end{pmatrix}}{\det \begin{pmatrix} v_M + iM_{ez} & iM_{ev} & -iM_{es} & -iM_{ea} \\ iM_{dz} & v_M + iM_{dv} & -iM_{ds} & -iM_{da} \\ iM_{fz} & iM_{fv} & v_M - iM_{fs} & -iM_{fa} \\ iM_{siz} & iM_{siv} & -iM_{sis} & v_M - iM_{sia} \end{pmatrix}}, \quad \mathcal{S}(q) = \pm 1 =$$



v_M and M_{ez} etc. are calculated as sums of ~ 1600 terms of powers of Θ, Ψ, $\cos(q)$, $\sin(q)$, and the roots w_a, w_b, and w_c.

The numerator of $A_d(q)$ is a sum of $\sim 1\,000\,000$ terms of powers of Θ, Ψ, $\cos(q)$, $\sin(q)$, and the roots w_a, w_b, and w_c.

still exact, but …

procedure

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Jordan Wigner representation of spin operators $T_{\cdots} = e^{\cdots c^\dagger_{\cdot\cdot} c_{\cdot\cdot}}$ c are Fermions, $\{c_{\cdot\cdot}, c^\dagger_{\cdot\cdot}\} = \delta_{\cdot\cdot,\cdot\cdot}$

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1. write T in Graßmann representation
2. calculate the eigenstate of T with highest eigenvalue: $\sim \boxed{e^{\gamma^*_i \mathcal{M}_{ij} \gamma^*_j}}$
3. use this eigenstate to calculate $\chi_{\square}$ (again, there are only quadratic forms of Graßmann variables)
4. take the resulting quotient of infinite dimensional matrices and write the result as Toeplitz-like finite dimensional determinant
5. beautify the integrands for the matrix elements ☜

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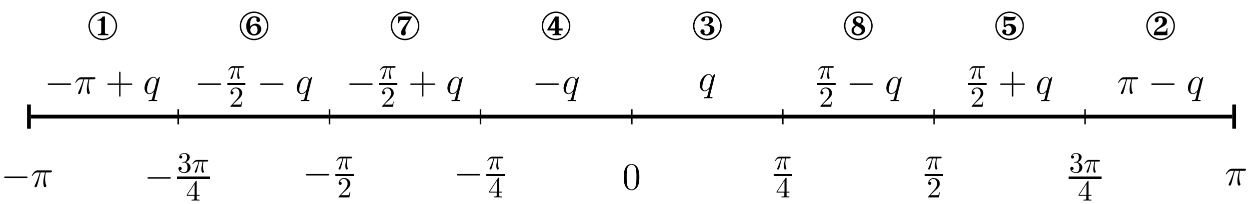
5. beautify the integrands for the matrix elements
6. calculate the matrix elements of the generalized Toeplitz matrix
7. determine the asymptotics of the generalized Toeplitz determinant

items (5), 6, and 7 are still to be performed numerically

technicalities

translation in a row is periodic with a period of 4

thus, in Fourier space, the interval $[-\pi, \pi]$ is divided up as



$\rightsquigarrow q \in [0, \frac{\pi}{4}]$ and 8×8 matrix structure

$T \sim e^H$ and H is formally a superconducting Hamiltonian with 8 states (as opposed to 2 states $c^\dagger_{k\uparrow},\ c^\dagger_{-k\downarrow}$)

- partial particle–hole transformation (for $\gamma_{②}, \gamma_{④}, \gamma_{⑥},$ and $\gamma_{⑧}$) $\rightsquigarrow$ normal Hamiltonian
- diagonalization
- back-transformation
- Fourier–back-transformation to real space $\rightsquigarrow$ $\boxed{e^{\gamma_i^* \mathcal{M}_{ij} \gamma_j^*}}$

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evaluation (computer algebra) is still being performed

Thank You !