Commutator identities on associative algebras and integrable equations in noncommuatative case

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Let we have some associative algebra, commuting elements $A_1, \ldots, A_n, A_1^{-1}, \ldots, A_n^{-1}$ (-1 is just a notation at the moment) belonging to this algebra. For an arbitrary element B of this algebra we introduce

$$S_B(A_1, \ldots, A_n) = A_1 A_2 B A_1^{-1} A_2^{-1}(A_1 - A_2) + (A_1 - A_{n-1}) A_n B A_n^{-1} + \text{cycle}(1, \ldots, n),$$

Because of commutativity of the set $A_1, \ldots, A_n, A_1^{-1}, \ldots, A_n^{-1}$ and associativity of the algebra we get that

$$S_B(A_1, A_2) = 0, \qquad S_B(A_1, A_2, A_3.A_4) = S_B(A_1, A_2, A_3) + S_B(A_1, A_3, A_4),$$

and so on for larger n. Now, assuming that associative algebra has unity and A_i^{-1} is inverse of A_i we get that also

$$S_B(A_1, A_2, A_3) \equiv A_1 A_2 B A_1^{-1} A_2^{-1}(A_1 - A_2) + (A_1 - A_2) A_3 B A_3^{-1} + \\ + \text{cycle}(1, 2, 3) = 0$$

again for any set of commuting elements A_1, A_2, A_3 and any element B. Then all $S_B(A_1, \ldots, A_n)$ with $n \geq 3$ also equal to zero consequently.

Let us denote

$$B(m) \equiv B(m_1, m_2, m_3) = \left(\prod_{n=1}^{3}(A - a_n)^{m_n}\right) B \left(\prod_{n=1}^{3}(A - a_n)^{m_n}\right)^{-1},$$

and let

$$B^{(1)}(m) = B(m_1 + 1, m_2, m_3), \quad B^{(2)}(m) = B(m_1, m_2 + 1, m_3), \quad \ldots,$$
$$B_i(m) = B^{(i)}(m) - B(m).$$

Then the identity means that function $B(m)$ obeys difference equation

$$B^{(12)}(A_1 - A_2) + (A_1 - A_2)B^{(3)} + \text{cycle}(1, 2, 3) = 0$$

Let B operator in $V \otimes W$, A operator in V, and a_1, a_2, a_3 be commuting operators in W. Choosing

$$A_i = A \otimes I - I \otimes a_i, \quad i = 1, 2, 3,$$

Then

$$B^{(12)}(a_1 - a_2) + (a_1 - a_2)B^{(3)} + \text{cycle}(1, 2, 3) = 0,$$

or

$$B_{12}(a_1 - a_2) + [a_1 - a_2, B_3] + \text{cycle} = 0,$$

Let we have (infinite) matrix $\mathcal{F} = \{\mathcal{F}_{m,n}\}_{m,n\in\mathbb{Z}}$. Any such matrix can be written in the form $\mathcal{F} = \sum_{n\in\mathbb{Z}} f_n \mathcal{T}^n$, where f_n are diagonal matrices $f_n = \mathrm{diag}\{\mathcal{F}_{m,m+n}\}_{m\in\mathbb{Z}}$ and $\mathcal{T}$ is operator of shift: $\mathcal{T}_{l.m} = \delta_{l+1,m}$. For any diagonal matrix $f = \mathrm{diag}\{f_m\}$ we have that $(\mathcal{T} f \mathcal{T}^{-1})_{lm} = f_{m-1}\delta_{lm}$. Product of two matrices can be written in the form $\mathcal{F}\,\mathcal{G} = \sum_n f_n \left(\sum_m \mathcal{T}^{n-m} g_m \mathcal{T}^{m-n}\right) T^n$.

Let us perform "shifted" Fourier transform: $F(\zeta,\zeta') = \sum\limits_{m,m'\in\mathbb{Z}} \zeta'^{m'-m}\zeta^{-m}\,\mathcal{F}_{m,m'}$, where $|\zeta|,|\zeta'| = 1$. Or $F(\zeta,\zeta') = \sum_n (\sum_m \zeta^{-m}(f_n)_m)\,\zeta'^n$. Next we formally continue this kernels in the complex domain with respect to ζ':

$$F(\zeta,\zeta') \longrightarrow F(\zeta;z), \quad z,\zeta\in\mathbb{C}, \quad |\zeta| = 1$$

Below we realize elements of the associative algebra as such functions (distributions) with composition law

$$(FG)(\zeta;z) = \oint_{|\zeta'|=1} \frac{d\zeta'}{2\pi i\zeta'} F(\zeta\overline{\zeta}';z\zeta')G(\zeta';z),$$

that in the matrix case is equivalent to the standard product of matrices.

For the unity matrix I we get

$$I(\zeta; z) = \delta_c(\zeta)$$

and for the shift operator $\mathcal{T}$:

$$T(\zeta; z) = z\delta_c(\zeta).$$

If $F(\zeta; z) = f(z)\delta_c(\zeta)$ then similarity transformation of an arbitrary operator G has kernel

$$(FGF^{-1})(\zeta; z) = \frac{f(\zeta z)}{f(z)} G(\zeta; z).$$

All standard operations on matrices can be reformulated in terms of their kernels. Say, kernel of Hermitian conjugation of F equals:

$$F^{\dagger}(\zeta; z) = \overline{F(\overline{\zeta}; \zeta/\overline{z})}.$$

But there appears a new operation:

$$(\overline{\partial} F)(\zeta; z) = \frac{\partial F(\zeta, z)}{\partial \overline{z}},$$

that is essential for the following construction.

Function $B(m)$ can be considered as function of two "space" variables (say, m_1 and m_2) and one "time" variable (correspondingly, m_3) with evolution given by the Hirota equation. We realize elements A and $B(m)$ of an associative algebra as extended operators in the above sense with kernels $A(\zeta; z)$ and $B(\zeta; z)$, that are operators in the auxiliary space W. We impose condition that

$$B(m_1, m_2, m_3; \zeta_1, \zeta_2; z) = \zeta_1^{m_1}\zeta_2^{m_2} B(m_3; \zeta_1, \zeta_2; z)$$

This gives two conditions:

$$B^{(1)} = (A - a_1)B(A - a_1)^{-1} = T_1 B T_1^{-1}, \quad B^{(2)} = (A - a_2)B(A - a_2)^{-1} = T_2 B T_2^{-1}$$

This means that we can choose $A = T_1 + a_1$, i.e. $A(\zeta; z) = (z + a_1)\delta_c(\zeta)$. Now the second condition takes the form

$$B^{(2)} = T_2 B T_2^{-1} = (T_1 + a_{12})B(T_1 + a_{12})^{-1}, \qquad a_{12} = a_1 - a_2.$$

i.e., there exists operator $L_0 = T_2 - T_1 + (a_2 - a_1)I$ such that $L_0 B = T_2 B T_2^{-1} L_0$, or $T_2 B(T_1 + a_{12}) = (T_1 + a_{12})BT_2$. As well $B^{(3)} = (T_1 + a_{13})B(T_1 + a_{13})^{-1}$. If $B(\zeta; z)$ is a matrix in the space W, and a_i are diagonal matrices in this space, then above relation means that $[z(\zeta_1 - \zeta_2) + a_{12,i} - \zeta_2 a_{12,j}]B_{ij}(\zeta; z) = 0$, or that there exists representation $B_{ij}(\zeta; z) = b_{ij}(\zeta)\delta(z(\zeta_1 - \zeta_2) + a_{12,i} - \zeta_2 a_{12,j})$.

We introduce operator ν with kernel $\nu(\zeta; z) = \nu(\zeta; z_1)$ as solution of the following d-bar problem:

$$\overline{\partial}\nu = \nu B, \qquad \lim_{z\to\infty} \nu(\zeta; z) = \delta_c(\zeta)$$

and assume its unique solvability. The m-dependence is introduced by $\overline{\partial}\nu(m) = \nu(m)B(m)$. In particular, $\overline{\partial}_1\nu^{(1)} = \nu^{(1)}T_1BT_1^{-1}$, or $\overline{\partial}_1(\nu^{(1)}T_1) = (\nu^{(1)}T_1)B$. Let us specify the $1/z_1$-term of expansion of $\nu(\zeta, z)$ at infinity:

$$\nu(m, \zeta; z) = \delta_c(\zeta) + \frac{u(m,\zeta)}{z_1} + \dots .$$

Then $\lim_{z_1\to\infty} \nu^{(1)T_1} = T_1 + u^{(1)}$. Then $\nu^{(1)} = T_1\nu T_1^{-1}$ and $u^{(1)} = T_1uT_1^{-1}$ (the r.h.s. is independent of z). In analogy: $\overline{\partial}_1(\nu^{(2)}(T_1 + a_{12})) = (\nu^{(1)}(T_1 + a_{12}))B$ and then

$$\nu^{(2)}(T_1 + a_{12}) = (T_1 + a_{12} + u^{(2)} - u^{(1)})\nu.$$

In the same way we derive $\nu^{(3)}(T_1 + a_{13}) = (T_1 + a_{13} + u^{(3)} - u^{(1)})\nu$.

We introduce:

$$\chi(m_1,m_2,m_3,z)=\oint\limits_{|\zeta_1|=1}\frac{d\zeta_1\,\zeta_1^{m_1-1}}{2\pi i}\oint\limits_{|\zeta_2|=1}\frac{d\zeta_2\,\zeta_2^{m_2-1}}{2\pi i}\nu(m_3,\zeta;z),$$

and $\varphi(m,z)=\chi(m,z)E(m,z)$, where $E(m,z)=z^{m_1}(z+a_{12})^{m_2}(z+a_{13})^{m_3}$. Then from above we get:

$$\begin{aligned}\varphi^{(2)}&=\varphi^{(1)}+\left(u^{(2)}-u^{(1)}+a_{12}\right)\varphi,\\ \varphi^{(3)}&=\varphi^{(2)}+\left(u^{(3)}-u^{(2)}+a_{23}\right)\varphi,\\ \varphi^{(1)}&=\varphi^{(3)}+\left(u^{(1)}-u^{(3)}+a_{31}\right)\varphi,\end{aligned}$$

so the Lax pair is any two of these equations. Compatibility condition gives:

$$u^{(12)}(u^{(2)}-u^{(1)}+a_{12})+a_{12}u^{(3)}+\text{cycle}=0$$

that is Hirota difference equation in noncommutative case. It is obvious nonlinearization of the original identity

$$B^{(12)}a_{12}+a_{12}B^{(3)}+\text{cycle}=0.$$

Let us denote

$$v(m) = u(m) - m_1a_1 - m_2a_2 - m_3a_3,$$

then all a_i are excluded from the equation and Lax pair:

$$\begin{aligned}
\varphi^{(2)} &= \varphi^{(1)} + \big(v^{(2)} - v^{(1)}\big)\varphi, \\
\varphi^{(3)} &= \varphi^{(2)} + \big(v^{(3)} - v^{(2)}\big)\varphi, \\
\varphi^{(1)} &= \varphi^{(3)} + \big(v^{(1)} - v^{(3)}\big)\varphi,
\end{aligned}$$

and

$$v^{(12)}(v^{(2)} - v^{(1)}) + \mathrm{cycle}(1,2,3) = 0,$$

while condition on asymptotics is essential ($u(m)$ is decaying).

Limiting cases. If we substitute $a_k \to x a_k$, where x is c-number, we get

$$B^{(k)} = a_k \left[B - \frac{1}{x} B_{t_k} \right] a_k^{-1} + \ldots, \quad x \to \infty$$

where $\partial_{t_k} = [A a_k^{-1}, \cdot]$.

Limit $a_3 \to \infty$. Let $k = 3$. Then $1/x$ term gives identity

$$B^{(12)} a_{12} + a_3 (B^{(2)} - B^{(1)})_{t_3} + a_2 B^{(1)} - a_1 B^{(2)} + \\ + a_3 B^{(2)} a_3^{-1} a_2 - a_3 B^{(1)} a_3^{-1} a_1 + a_{12} a_3 B a_3^{-1} = 0.$$

Thus,

$$B^{(1)} = T_1 B T_1^{-1}, \quad B^{(2)} = (T_1 + a_{12}) B (T_1 + a_{12})^{-1}, \quad B_{t_3} = [(T_1 + a_1) a_3^{-1}, B],$$

so that $\overline{\partial}(\nu_{t_3} + \nu (T_1 + a_1) a_3^{-1}) = (\nu_{t_3} + \nu (T_1 + a_1) a_3^{-1}) B$. Thus again taking asymptotic into account we derive: $\nu_{t_3} + \nu (T_1 + a_1) a_3^{-1} = a_3^{-1} (T_1 + a_3 u a_3^{-1} - u^{(1)} + a_1) \nu$. Finally for $w(m_1, m_2, t_3) = u(m_1, m_2, t_3) - m_1 a_1 - m_2 a_2$ we get Lax pair and evolution equation

$$\psi_{t_3} = \psi^{(1)} - w_1 \psi,$$
$$\psi^{(2)} = \psi^{(1)} + (w_2 - w_1) \psi,$$
$$(w_2 - w_1)_{t_3} + w_{12} (w_2 - w_1) + [w_1, w_2] = 0.$$

Limit $a_2 \to \infty$. The identity takes the form:

$$(a_2 B a_2^{-1} - a_3 B a_3^{-1})^{(1)} a_1 + (a_2 B_{t_2} - a_3 B_{t_3})^{(1)} - \\ - a_2 a_3 B_{t_2} a_3^{-1} + a_3 a_2 B_{t_3} a_2^{-1} - a_1 a_2 B a_2^{-1} + a_1 a_3 B a_3^{-1} = 0,$$

that is antisymmetric with respect to indexes 2 and 3. Now we have

$$B^{(1)} = T_1 B T_1^{-1}, \quad B_{t_2} = [(T_1 + a_1) a_2^{-1}, B], \quad B_{t_3} = [(T_1 + a_1) a_3^{-1}, B],$$

Substitution: $v(m_1, t_2, t_3) = u(m_1, t_2, t_3) - m_1 a$, Lax pair

$$\alpha \psi_{t_2} = \psi^{(1)} + [\alpha w \alpha^{-1} - w^{(1)}]\psi, \\ \alpha^{-1} \psi_{t_3} = \psi^{(1)} + [\alpha^{-1} w \alpha - w^{(1)}]\psi,$$

and equation:

$$\left(w\alpha - \alpha w^{(1)}\right)_{t_2} - \left(w\alpha^{-1} - \alpha^{-1} w^{(1)}\right)_{t_3} + [w\alpha - \alpha w^{(1)}, w\alpha^{-1} - \alpha^{-1} w^{(1)}] = 0.$$

where a and α are constant, mutually commuting matrices.

Limit $a_1 \to \infty$. Limiting identity reads as

$$a_1 \partial_{t_1} (a_2 B a_2^{-1} - a_3 B a_3^{-1}) + \text{cycle} = 0.$$

we get that the Lax pair is any two equations of the system

$$\begin{aligned} a_1\, \varphi_{t_1} &= a_2\, \varphi_{t_2} + (a_1 u a_1^{-1} - a_2 u a_2^{-1})\, \varphi, \\ a_2\, \varphi_{t_2} &= a_3\, \varphi_{t_3} + (a_2 u a_2^{-1} - a_3 u a_3^{-1})\, \varphi, \\ a_3\, \varphi_{t_3} &= a_1\, \varphi_{t_1} + (a_3 u a_3^{-1} - a_1 u a_1^{-1})\, \varphi, \end{aligned}$$

and equation of compatibility is

$$a_1 \big(a_3 u a_3^{-1} - a_2 u a_2^{-1}\big)_{t_1} + a_2 a_3 u a_2^{-1} a_3^{-1} \big(a_3 u a_3^{-1} - a_2 u a_2^{-1}\big) + \text{cycle}(1,2,3) = 0.$$

Limits of equal a_i. We write $a_j = a_i + xb_{ij}$, where b_{ij} is some operator (commuting with all a_k) and x is a parameter, $x \to 0$. Then

$$(A-a_j)B(A-a_j)^{-1} \to (A-a_i)\big(B - x[b_{ij}(A-a_i)^{-1}, B] + o(x)\big)(A-a_i)^{-1},$$

that means that we can introduce, say, t_{ij} by means of $\partial_{t_{ij}} B = [b_{ij}(A-a_i)^{-1}, B]$. Then

$$B^{(j)} \to B^{(i)} - xB^{(i)}_{t_{ij}} + o(x).$$

Limit $a_3 \to a_1$. Identity in the first order on x gives

$$B^{(12)}_{t_3} a_{12} - a_{12} B^{(1)}_{t_3} - \big(B^{(12)} - B^{(11)}\big)b_3 + b_3\big(B^{(2)} - B^{(1)}\big) = 0,$$

where

$$B^{(1)} = T_1 B T_1^{-1}, \quad B^{(2)} = (T_1 + a_{12})B(T_1 + a_{12})^{-1}, \quad B_{t_3} = [b_3 T_1^{-1}, B].$$

Introducing $v(m_1, m_2, t_3) = u(m_1, m_2, t_3) - a_1 m_1 - a_2 m_2 + b_3 t_3$ we get finally

$$\varphi^{(2)} = \varphi^{(1)} + (v_2 - v_1)\,\varphi,$$
$$\varphi_{t_3} = v_{t_3}\,\varphi^{(-1)}\,.$$

and equation:

$$(v^{(2)} - v^{(1)})^{(1)} v^{(1)}_{t_3} - v^{(12)}_{t_3}(v^{(2)} - v^{(1)}) = 0.$$

Limit $a_2 \to a_1$. We set now $a_2 = a_1 + x b_2$ and consider limit $x \to 0$. Identity takes the form

$$\left(B^{(1)} b_3 - b_3 B\right)_{t_2} = \left(B^{(1)} b_2 - b_2 B\right)_{t_3},$$

$$B(m_1, t_2, t_3) = T_1^{m_1} e^{(t_2 b_2 + t_3 b_3) T_1^{-1}} B T_1^{-m_1} e^{-(t_2 b_2 + t_3 b_3) T_1^{-1}},$$

We introduce $v(m_1, t_2, t_3) = u(m_1, t_2, t_3) - a_1 m_1 + b_2 t_2 + b_3 t_3$, Lax pair and equation:

$$\varphi_{t_2} = v_{t_2} \varphi^{(-1)},$$
$$\varphi_{t_3} = v_{t_3} \varphi^{(-1)},$$
$$v_{t_2}^{(1)} v_{t_3} = v_{t_3}^{(1)} v_{t_2}.$$

Limit $a_3 \to \infty$ and $a_2 \to a_1$. Noncummutative Toda chain. Identity:

$$B^{(1)}b_2 + a_3 B_{t_2t_3} - b_2 B - a_1 B_{t_2} + a_3 B_{t_2} a_3^{-1} a_1 -$$
$$- a_3 B a_3^{-1} b_2 + b_2 a_3 B^{(-1)} a_3^{-1} = 0.$$

Correspondingly,

$$B^{(1)} = T_1 B T_1^{-1}, \quad B_{t_2} = [b_2 T_1^{-1}, B], \quad B_{t_3} = [(T_1 + a_1) a_3^{-1}, B].$$

Lax pair and equation:

$$\psi_{t_2} = w_{t_2} \psi^{(-1)},$$
$$\psi_{t_3} = \psi^{(1)} - w_1 \psi,$$
$$w_{t_2t_3} + w_{t_2} w_{-1} + w_1 w_{t_2} = 0,$$

where asymptotically $w(m_1, t_2, t_3) \to -m_1 a_1 a_3^{-1} + b_2 a_3^{-1} t_2$. In the commutative case $w_{t_2} = b_2 a_3^{-1} e^{\phi_{-1}}$, and $\phi_{t_2t_3} = a_1 a_3^{-1}\big(e^{-\phi_1} - e^{\phi_{-1}}\big)$.