

# Polynomial forms for integrable quantum Hamiltonians of second order with two independent variables

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# 1. Introduction

Consider integrable Hamiltonians

$$H = \Delta + U(x_1, \dots, x_n)$$

related to simple Lie algebras. For such Hamiltonians the potential  $U$  is a rational, trigonometric or elliptic function. For instance, the elliptic Calogero-Moser Hamiltonian is given by

$$H = \Delta + g \sum_{i>j} \wp(x_i - x_j).$$

**Observation 1** (A.Turbiner). Some of these Hamiltonians admits a change of variables that bring it to a differential operator with polynomial coefficients.

**Example.** Consider the Calogero model with  $n = 3$ :

$$H = \Delta + g \sum_{i>j}^3 \frac{1}{(x_i - x_j)^2}.$$

Let  $Y = \sum_{i=1}^3 x_i$  and  $y_i = x_i - \frac{Y}{3}$ . Then

$$\Delta = -3 \frac{\partial^2}{\partial Y^2} - \frac{2}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2} \right).$$

Thus we have reduced the Hamiltonian to the following two dimensional one:

$$\mathcal{H} = -\frac{1}{3} \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} - \frac{\partial^2}{\partial y_1 \partial y_2} \right) + \nu(\nu - 1) \sum_{i>j}^3 \frac{1}{(y_i - y_j)^2}. \quad (1)$$

Here  $y_3 = -y_1 - y_2$ .

The transformation

$$x = -y_1^2 - y_2^2 - y_1 y_2, \quad y = -y_1 y_2 (y_1 + y_2)$$

brings  $\mathcal{H}$  to the polynomial form

$$L = -2x \frac{\partial^2}{\partial x^2} - 6y \frac{\partial^2}{\partial x \partial y} + \frac{2}{3} x^2 \frac{\partial^2}{\partial y^2} - 2(1 + 3\nu) \frac{\partial}{\partial x}. \quad \square$$

In the trigonometric case the transformation to a polynomial form is given by

$$x = \cos y_1 + \cos y_2 + \cos (y_1 + y_2) - 3,$$

$$y = \sin y_1 + \sin y_2 - \sin (y_1 + y_2).$$

**Theorem.** The transformation

$$x = \frac{\wp'(y_1) - \wp'(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}, \quad y = \frac{\wp(y_1) - \wp(y_2)}{\wp(y_1)\wp'(y_2) - \wp(y_2)\wp'(y_1)}$$

brings the elliptic Calogero-Moser Hamiltonian to a polynomial form.

## 2. Classification of the polynomial forms

Consider second order differential operators

$$L = a(x, y) \frac{\partial^2}{\partial x^2} + 2b(x, y) \frac{\partial^2}{\partial x \partial y} + c(x, y) \frac{\partial^2}{\partial y^2} + d(x, y) \frac{\partial}{\partial x} + e(x, y) \frac{\partial}{\partial y} + f(x, y) \quad (2)$$

with polynomial coefficients. Denote by  $D(x, y)$  the determinant  $a(x, y)c(x, y) - b(x, y)^2$ .

The operators we are interested in should have three important properties:

**1.** We assume that the associated contravariant metric

$$g^{1,1} = a, \quad g^{1,2} = g^{2,1} = b, \quad g^{2,2} = c,$$

is flat.

This is equivalent to

$$R_{1,2,1,2} = ab^2 b_{xx} + \cdots + c^2 d a_y = 0.$$

**Example.** For any constant  $\alpha, \beta, \gamma$  the metric  $g$  with

$$a = x^3 - 3xy + \alpha(x^2 - 2y) + \beta x + 2\gamma,$$

$$b = x^2 y - 2y^2 + \alpha xy + 2\beta y + \gamma x,$$

$$c = xy^2 + 2\alpha y^2 + \beta xy + \gamma(x^2 - 2y)$$

is flat.

**2.** The operator should be potential. This means that

$$\begin{aligned} & \frac{\partial}{\partial y} \left( \frac{be - cd + c(a_x + b_y) - b(b_x + c_y)}{D} \right) \\ &= \frac{\partial}{\partial x} \left( \frac{bd - ae + a(b_x + c_y) - b(a_x + b_y)}{D} \right). \end{aligned} \quad (3)$$

The properties 1 and 2 guaranty that  $L$  can be reduced to the form

$$\bar{L} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + V(x, y)$$

by a proper change of variables.

**3.** (A. Turbiner) The operator has to preserve the vector space of all polynomials  $P(x, y)$  such that  $\deg P \leq n$  for some  $n > 1$ .



If  $L$  satisfies Property 3 then the coefficients of  $L$  have the following structure

$$a = q_1x^4 + q_2x^3y + q_3x^2y^2 + k_1x^3 + k_2x^2y + k_3xy^2 + \\ a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6;$$

$$b = q_1x^3y + q_2x^2y^2 + q_3xy^3 + \frac{1}{2} \left( k_4x^3 + (k_1 + k_5)x^2y + (k_2 + k_6)xy^2 + k_3y^3 \right) + \\ b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6;$$

$$c = q_1x^2y^2 + q_2xy^3 + q_3y^4 + k_4x^2y + k_5xy^2 + k_6y^3 + \\ c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6;$$

$$d = (1-n) \left( 2(q_1x^3 + q_2x^2y + q_3xy^2) + k_7x^2 + (k_2 + k_8 - k_6)xy + k_3y^2 \right) +$$

$$d_1x + d_2y + d_3;$$

$$e = (1-n) \left( 2(q_1x^2y + q_2xy^2 + q_3y^3) + k_4x^2 + (k_5 + k_7 - k_1)xy + k_8y^2 \right) +$$

$$e_1x + e_2y + e_3;$$

$$f = n(n-1) \left( q_1x^2 + q_2xy + q_3y^2 + (k_7 - k_1)x + (k_8 - k_6)y \right) + f_1.$$

The dimension of the space of such operators equals 36.

The group  $GL_3$  acts on this vector space by the formula

$$\tilde{x} = \frac{P}{R}, \quad \tilde{y} = \frac{Q}{R}, \quad \tilde{L} = R^{-n}LR^n,$$

where  $P, Q, R$  are polynomials of degree one in  $x$  and  $y$ .

This representation is a sum of irreducible representations  $V_1$ ,  $V_2$  and  $V_3$  of dimensions 27, 8 and 1 correspondingly. A basis in  $V_2$  is given by

$$x_1 = 5k_7 - k_5 - 7k_1, \quad x_2 = 5k_8 - k_2 - 7k_6,$$

$$x_3 = 5d_1 + 2(n-1)(2a_1 + b_2), \quad x_4 = 5e_1 + 2(n-1)(2b_1 + c_2),$$

$$x_5 = 5d_2 + 2(n-1)(2b_3 + a_2), \quad x_6 = 5e_2 + 2(n-1)(2c_3 + b_2),$$

$$x_7 = 5d_3 + 2(n-1)(a_4 + b_5), \quad x_8 = 5e_3 + 2(n-1)(b_4 + c_5).$$

The generic orbit of the action on  $V_2$  has dimension 6. There are two polynomial invariants of the action:

$$I_1 = x_3^2 - x_3x_6 + x_6^2 + 3x_4x_5 + 3(n-1)(x_1x_7 + x_2x_8),$$

and

$$I_2 = 2x_3^3 - 3x_3^2x_6 - 3x_3x_6^2 + 2x_6^3 + 9x_4x_5(x_3 + x_6) +$$

$$9(n-1)(x_1x_3x_7 + x_2x_6x_8 - 2x_1x_6x_7 - 2x_2x_3x_8 + 3x_2x_4x_7 + 3x_1x_5x_8).$$

## Flat potential operators with discrete symmetries

For almost all known examples the operator  $L$  possesses additional finite group of discrete symmetries. For example, the operator with coefficients

$$\begin{aligned}a &= x^2(x^2 + y^2) + \alpha x^2 + \beta y^2, & b &= xy(x^2 + y^2) + (\alpha - \beta)xy, \\c &= y^2(x^2 + y^2) + \beta x^2 + \alpha y^2, & d &= 2(n - 1)x(\lambda - x^2 - y^2), \\e &= 2(n - 1)y(\lambda - x^2 - y^2), & f &= n(n - 1)(x^2 + y^2).\end{aligned}$$

is flat, potential, and possesses the discrete group of symmetries  $D_4$ , generated by reflections

$$x \rightarrow -x, y \rightarrow y, \quad x \rightarrow x, y \rightarrow -y, \quad x \rightarrow y, y \rightarrow x.$$

Consider the case when  $L$  is invariant with respect to a reflection. Using a transformation, we reduce the reflection to the form  $\tilde{x} = x$ ,  $\tilde{y} = -y$ . The coefficients of the operator  $L$  have the following symmetry properties:

$$\begin{aligned} a(x, -y) &= a(x, y), & b(x, -y) &= -b(x, y), & c(x, -y) &= c(x, y), \\ d(x, -y) &= d(x, y), & e(x, -y) &= -e(x, y), & f(x, -y) &= f(x, y). \end{aligned}$$

The class of such operators allows the transformation group

$$\tilde{x} = \frac{\alpha x + \beta}{\gamma x + \delta}, \quad \tilde{y} = \frac{y}{\gamma x + \delta}. \quad (4)$$

Transformations  $\tilde{L} = c_1 L + c_2$  are also allowed.

The coefficients  $a, b$  and  $c$  can be written in the form

$$a = P + Qy^2, \quad b = \frac{1}{4}(P' - R)y + \frac{1}{2}Q'y^3,$$

$$c = \left(S + \frac{1}{12}P'' - \frac{1}{4}R' + \sigma\right)y^2 + \frac{1}{2}Q''y^4.$$

where  $\deg P = 4$ ,  $\deg Q = \deg R = \deg S = 2$ .

Under transformations (4) the polynomial  $P$  changes as follows

$$\tilde{P} = (\gamma x + \delta)^4 P\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right). \quad (5)$$

**Definition.** A differential operator  $L$  is called **elliptic** if the polynomial  $P$  has four different roots on the Riemann sphere. It is called **trigonometric** if  $P$  has one double root.

## Classification of the elliptic models

In the elliptic case without loss of generality we set

$$P(x) = x(x-1)(x-u).$$

**Proposition 1.** If the property 1 holds then any root of the polynomial  $S$  is a root of the polynomial  $P$ .  $\square$

It follows from Proposition 1 that there are two alternatives: **A:**  $S = kx^2$  and **B:**  $S = kx(x-1)$ .

**Theorem 1.** In Case **A** we obtain from  $R_{1,2,1,2} = 0$  that

$$S(x) = x^2, \quad R(x) = -\frac{5}{3}(x^2 - 2x + 3u - 2ux),$$

$$Q(x) = \frac{1}{9}(x^2 - x + 1 + u^2 - ux - u), \quad \sigma = 0.$$

It follows from (3) that

$$d = \frac{1}{9}(1-n) \left( 3(5x^2 - 4x - 4ux + 3u) + (2x - 1 - u)y^2 \right),$$

$$e = \frac{2}{9}(1-n)y \left( 9x + y^2 - 6u - 6 \right), \quad f = \frac{1}{9}n(n-1) \left( 6x + y^2 \right). \quad \square$$

**Theorem 2.** In Case **B**

$$S(x) = x(x - 1), \quad R(x) = -3(x^2 - 2ux + u),$$

$$Q(x) = \frac{1}{2}(x^2 - 2ux + 2u^2 - u), \quad \sigma = \frac{1}{3}(2u - 1),$$

$$d = (1 - n) \left( 2\lambda_1(x - u) + \lambda_2(x^2 - x + 2ux - 4u^2 + 2u) + \right. \\ \left. 2(1 - u)(x - 2u) + (x - u)y^2 \right),$$

$$e = (1 - n) \left( \lambda_1 y + \lambda_2 xy + xy + y^3 \right),$$

$$f = n(n - 1) \left( \lambda_2 x - x + \frac{1}{2}y^2 \right).$$