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Manakov-Santini hierarchy: geometry and reductions

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Outline

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2. Application: Einstein-Weil geometry in 3D
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The Manakov-Santini system

The Manakov-Santini system – two-component integrable extension of the dKP equation,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_{xt} &= v_{yy} + u v_{xx} + v_x v_{xy} - v_{xx} v_y. \end{aligned}$$

Lax pair

$$\begin{aligned} \partial_y \boldsymbol{\Psi} &= ((p - v_x)\partial_x - u_x \partial_p)\boldsymbol{\Psi}, \\ \partial_t \boldsymbol{\Psi} &= ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p)\boldsymbol{\Psi}, \end{aligned}$$

where p plays a role of a spectral variable. For $v = 0$ reduces to dKP (Khohlov-Zabolotskaya equation)

$$u_{xt} = u_{yy} + (uu_x)_x,$$

reduction $u = 0$ gives the equation (Pavlov, Martinez Alonso and Shabat)

$$v_{xt} = v_{yy} + v_x v_{xy} - v_{xx} v_y.$$

Application: Einstein-Weil structures

M. Dunajski, E.V. Ferapontov and B. Kruglikov, On the Einstein-Weyl and conformal self-duality equations, arXiv:1406.0018v1

EW geometry on a three-dimensional manifold M^3 consists of a conformal structure $[g]$ and a symmetric connection $\mathbb{D}$ compatible with $[g]$ in the sense that, for any $g \in [g]$,

$$\mathbb{D}g = \omega \otimes g$$

for some covector ω, and such that the trace-free part of the symmetrized Ricci tensor of $\mathbb{D}$ vanishes. In coordinates, this gives

$$\mathbb{D}_k g_{ij} = \omega_k g_{ij}, \quad R_{(ij)} = \Lambda g_{ij},$$

where $\omega = \omega_k dx^k$ is a covector, $R_{(ij)}$ is the symmetrized Ricci tensor of $\mathbb{D}$, and Λ is some function. In fact one needs to specify g and ω only, then the first set of equations uniquely defines $\mathbb{D}$.

The Einstein-Weyl equations are integrable by twistor-theoretic methods.

Theorem

There exists a local coordinate system (x, y, t) *on* M^3 *such that any Lorentzian Einstein–Weyl structure is locally of the form*

$$g = -(dy - v_x dt)^2 + 4(dx - (u - v_y)dt)dt,$$
$$\omega = -v_{xx}dy + (4u_x - 2v_{xy} + v_x v_{xx})dt,$$

where the functions u *and* v *on* M^3 *satisfy a coupled system of second-order PDEs (MS system)*

$$P(u) + u_x^2 = 0, \quad P(v) = 0,$$

where

$$P = \partial_x \partial_t - \partial_y^2 + (u - v_y)\partial_x{}^2 + v_x \partial_x \partial_y.$$

Two-component extension of dispersionless d2DTL system

L.V. Bogdanov, JPA 43 (2010) 434008

$$(\mathrm{e}^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{ty},$$
$$m_{tt}\mathrm{e}^{-\phi} = m_{ty}m_x - m_{xy}m_t.$$

The Lax pair

$$\partial_x \mathbf{\Psi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t \frac{m_x}{m_t} - \phi_x)\partial_\lambda\right)\mathbf{\Psi},$$
$$\partial_y \mathbf{\Psi} = \left(-\frac{1}{\lambda}\frac{\mathrm{e}^{-\phi}}{m_t}\partial_t - \frac{(\mathrm{e}^{-\phi})_t}{m_t}\partial_\lambda\right)\mathbf{\Psi}$$

For $m = t$ the system reduces to the dispersionless 2DTL equation

$$(\mathrm{e}^{-\phi})_{tt} = \phi_{xy},$$

The reduction $\phi = 0$ (Pavlov; Shabat and Martinez Alonso)

$$m_{tt} = m_{ty}m_x - m_{xy}m_t.$$

Also gives generic Einstein-Weil structures, a symmetric complex version $(x, y \to z, \bar{z})$ is probably suitable for Euclidian case.

Differential geometry and the hierarchy. General scheme

Bogdanov and Konopelchenko, JPA 2013, Journal of Physics Conf. Series 2014

A pair of commuting holomorphic (in spectral variable) vector fields can be equivalently described in terms of holomorphic decomposable (Plüker) form Ω_m (m+2 variables). Commutativity is equivalent to gauge-invariant closedness of the form, defined as existence of $J(\lambda, \mathbf{x})$ such that $d(J\,\Omega_m) = 0$.

In general, (gauge invariantly) closed Plüker form Ω_m holomorphic with respect to $\lambda = x_0$ defines a hierarchy in terms of commuting vector fields holomorphic in λ (integrable distribution), the equations of the hierarchy are given by the gauge-invariant closedness equations.

For the Manakov-Santini system we consider decomposable 2-form with polynomial coefficients

$$\begin{aligned}
\Omega_2 \quad = \quad & d\lambda \wedge dx_1 - a_{11} d\lambda \wedge dx_2 - a_{21} d\lambda \wedge dx_3 + a_{10} dx_1 \wedge dx_2 + \\
& + a_{20} dx_1 \wedge dx_3 - (a_{11} a_{20} - a_{10} a_{21}) dx_2 \wedge dx_3 \\
& a_{10} = u_0(x), \quad a_{11} = u_1(x) + \lambda, \\
& a_{20} = v_0(x) + \lambda v_1(x), \quad a_{21} = v_2(x) + \lambda v_3(x) + \lambda^2.
\end{aligned}$$

Denoting $x = x_1$, $y = x_2$, $t = x_3$, $u_0 = u_x$, $u_1 = -v_x$, from gauge invariant closedness equations one gets the Manakov-Santini system

$$\begin{aligned}
u_{xt} + u_{yy} + u_x^2 + (u - v_y) u_{xx} + v_x u_{xy} = 0, \\
v_{xt} + v_{yy} + v_x v_{xy} + (u - v_y) v_{xx} = 0.
\end{aligned}$$

Reductions

1.The form Ω_2 is closed in the standard sense - dKP (Hamiltonian vector fields). In general, the closedness leads to volume-preserving vector fields.

2.Reduction $\Omega_2 \wedge d\lambda = 0$. In this case it is possible to consider Ω_1 not containing $d\lambda$, $\Omega_2 = \Omega_1 \wedge d\lambda$. Vector fields do not contain a derivative over spectral variable. Leeds to Pavlov system and Martinez Alonso - Shabat universal hierarchy, for general Ω_m - to HCR hierarchies.

The MS hierarchy in terms of differential form

Considering more variables and higher order polynomials in the form Ω_2, we come to MS hierarchy.

General properties of the form Ω_2:

1. $\Omega_2 \wedge \Omega_2 = 0$ (Decomposability for 2-forms, equivalent to Plücker relations)
2. $\exists J: \ d(J\,\Omega_2) = 0$ (gauge-invariant closedness)

For decomposable forms the second condition is equivalent to the set of equations for the coefficients of the form, coinciding with involutiveness equations for the distribution defined by the form.

The form Ω_2 possessing these two properties can be represented as

$$\Omega_2 = J^{-1} dL \wedge dM,$$

J, L, M - some series in spectral variable. Polynomiality of coefficients of the form leads to the condition

$$(J^{-1} dL \wedge dM)_- = 0.$$

For the case

$$\Omega_2 = dp \wedge dx + \dots$$

(p - spectral variable) Ω_2 is represented as

$$\Omega_2 = \{L, M\}^{-1} dL \wedge dM$$

Generating relation for MS hierarchy is

$$(\{L, M\}^{-1} dL \wedge dM)_{-} = 0,$$

considered for the series of the form

$$L = p + \sum_{n=1}^{\infty} u_n(\mathbf{t}) p^{-n},$$

$$M = M_0 + M_1, \quad M_0 = \sum_{n=0}^{\infty} t_n L^n,$$

$$M_1 = \sum_{n=1}^{\infty} v_n(\mathbf{t}) L^{-n} = \sum_{n=1}^{\infty} \tilde{v}_n(\mathbf{t}) p^{-n},$$

Lax-Sato equations of MS hierarchy

Generating relation implies Lax-Sato equations

$$\frac{\partial}{\partial t_n}\begin{pmatrix} L \\ M \end{pmatrix} = \left(\left(\frac{L^n L_p}{\{L, M\}}\right)_+ \partial_x - \left(\frac{L^n L_x}{\{L, M\}}\right)_+ \partial_p\right)\begin{pmatrix} L \\ M \end{pmatrix},$$

Lax-Sato equations for the first two flows of the hierarchy

$$\partial_y \begin{pmatrix} L \\ M \end{pmatrix} = ((p - v_x)\partial_x - u_x\partial_p)\begin{pmatrix} L \\ M \end{pmatrix}$$
$$\partial_t \begin{pmatrix} L \\ M \end{pmatrix} = ((p^2 - v_x p + u - v_y)\partial_x - (u_x p + u_y)\partial_p)\begin{pmatrix} L \\ M \end{pmatrix}$$

where $u = u_1$, $v = v_1$, $x = t_0$, $y = t_1$, $t = t_2$, correspond to the Lax pair of the Manakov-Santini system

Dressing scheme

Let us consider the form Ω_2

$$\Omega_2 = \{L, M\}^{-1} dL \wedge dM$$

Question

How to provide analyticity of the form Ω_2? What kind of functions L, M in the complex plane correspond to a polynomial form?

It is easy to see that Ω_2 is invariant under diffeomorphism

$$(L, M) \to \mathbf{F}(L, M)$$

Let L, M be holomorphic outside some curve in the complex plain, having a discontinuity on it. If they satisfy a nonlinear vector Riemann-Hilbert problem (nvRHp)

$$(L, M)_{\text{in}} = \mathbf{F}(L, M)_{\text{out}},$$

then the form Ω_2 is holomorhic in all the complex plane.

It is also possible to introduce nonanalyticity in some domain G,

$$\bar{\partial}(L, M) = \mathbf{f}(L, M),$$

it is easy to check that Ω_2 is analytic in G.

Nonlinear vector Riemann-Hilbert and $\bar{\partial}$ problems give a tool to construct Ω_2 with polynomial (holomorphic) coefficients, generating commuting vector fields with polynomial coefficients and a solution to MS hierarchy

The reduction to dKP hierarchy corresponds to area-preserving diffeomorpism in nonlinear vector Riemann-Hilbert problem (respectively divergence-free $\mathbf{f}$ in the $\bar{\partial}$-problem), in this case the form

$$\tilde{\Omega}_2 = dL \wedge dM$$

is analytic and

$$\{L, M\} = 1$$

Dunajski interpolating system

The condition used by Dunajski (JPA 2008) to reduce the Manakov-Santini system to the interpolating system

$$\alpha u = v_x,$$

The reduced MS system can be written as deformed dKP,

$$\begin{aligned} u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\ v_x &= \alpha u, \end{aligned}$$

it also implies a single equation for v,

$$v_{xt} = v_{yy} + \alpha^{-1} v v_{xx} + v_x v_{xy} - v_{xx} v_y.$$

The limit $\alpha \to 0$ corresponds to dKP, $\alpha \to \infty$ – to equation, introduced by Pavlov, Martinez Alonso and Shabat
Dunajski interpolating system describes "a symmetry reduction of the anti-self-dual Einstein equations in (2, 2) signature by a conformal Killing vector whose selfdual derivative is null".

A class of differential reductions of the MS hierarchy

The dynamics of the Poisson bracket $J = \{L, M\}$, $J = 1 + v_x p^{-1} + \dots$ is described by the nonhomogeneous equation

$$\frac{\partial}{\partial t_n} \ln J = (A_n \partial_x - B_n \partial_p) \ln J + \partial_x A_n - \partial_p B_n,$$

$$A_n = \left(\frac{L^n L_p}{J}\right)_+, \quad B_n = \left(\frac{L^n L_x}{J}\right)_+,$$

A_n, B_n are polynomials in p. $\ln J + F(L, M)$ also satisfies these equations. We define a class of reductions of Manakov-Santini hierarchy by the condition

$$(\ln J - \alpha L^k)_- = 0,$$

where α is a constant. Then $\ln J - \alpha L^k$ is a polynomial,

$$\{L, M\} = \exp \alpha (L^k - L^k{}_+).$$

Characterization of the reduction

Proposition

The existence of a polynomial solution

$$f = -\alpha p^k + \sum_{0}^{i=k-2} f_i(\mathbf{t}) p^i,$$

(where the coefficients f_i don't contain constants, see below) of equations

$$\frac{\partial}{\partial t_n} f = (A_n \partial_x - B_n \partial_p) f + \partial_x A_n - \partial_p B_n,$$

is equivalent to the reduction condition

$$(\ln J - \alpha L^k)_- = 0,$$

General k

$$(\ln J - \alpha L^k)_- = 0 \Rightarrow (\ln J - \alpha L^k) = (\ln J - \alpha L^k)_+ = -\alpha (L^k)_+,$$

$f = -\alpha (L^k)_+$ is a solution of nonhomogeneous equation of the Proposition.

$$J = \exp \alpha (L^k - (L^k{}_+)) = \exp \alpha (L^k{}_-),$$

and Lax-Sato equations of reduced hierarchy read

$$\frac{\partial}{\partial t_n} L = (e^{-\alpha (L^k{}_-)} L^n L_p)_+ \partial_x L - (e^{-\alpha (L^k{}_-)} L^n L_x)_+ \partial_p L.$$

Generating relation takes the form

$$\left(e^{-\alpha L^k} \mathrm{d}L \wedge \mathrm{d}M \right)_- = 0.$$

For the first flow $n = 1$ we obtain a condition

$$\partial_y (\alpha L^k{}_+) = ((p - v_x)\partial_x - u_x \partial_p)(\alpha L^k{}_+) + v_{xx}.$$

This condition defines a differential reduction of Manakov-Santini system.

The case $k = 0$ (or $\alpha = 0$) corresponds to Hamiltonian vector fields. Indeed, in this case $J = 1$, and from nonhomogeneous equations we have

$$\partial_x A_n - \partial_p B_n = 0.$$

This is the case of the dKP hierarchy.

Proposition

The reduction with general k is 'interpolating' between the dKP hierarchy ($\alpha \to 0$), and the Gelfand-Dikii reduction of the MS hierarchy of the order k, $L_-^k = 0$, for $\alpha \to \infty$.

(directly follows from the definition of the reduction)

$k = 1$. Dunajski interpolating system

In the case $k = 1$

$$(\ln J - \alpha L)_- = 0 \Rightarrow (\ln J - \alpha L) = (\ln J - \alpha L)_+ = -\alpha p,$$
$$J = \exp \alpha (L - p).$$

Lax-Sato equations

$$\frac{\partial}{\partial t_n} L = (e^{\alpha(p-L)} L^n L_p)_+ \partial_x L - (e^{\alpha(p-L)} L^n L_x)_+ \partial_p L.$$

The generating relation for the reduced hierarchy reads

$$\left(e^{\alpha(p-L)} \mathrm{d}L \wedge \mathrm{d}M \right)_- = 0 \Rightarrow \left(e^{-\alpha L} \mathrm{d}L \wedge \mathrm{d}M \right)_- = 0.$$

Differential reduction reads

$$\alpha u = v_x,$$

which is exactly the condition used by Dunajski (JPA 2008) to reduce the Manakov-Santini system to the interpolating system.

The reduced MS system (equivalent to Dunajski interpolating system) can be written as deformed dKP,

$$
\begin{aligned}
u_{xt} &= u_{yy} + (uu_x)_x + v_x u_{xy} - u_{xx} v_y, \\
v_x &= \alpha u,
\end{aligned}
$$

it also implies a single equation for v,

$$
v_{xt} = v_{yy} + \alpha^{-1} v v_{xx} + v_x v_{xy} - v_{xx} v_y .
$$

The limit $\alpha \to 0$ corresponds to dKP, $\alpha \to \infty$ – to equation, introduced by Pavlov.

Differential reductions. Special cases

The case $k = 2$.

$$J = e^{\alpha(L^2{}_-)}$$

Differential reduction for the MS system

$$2\alpha(u_y + v_x u_x) = v_{xx}$$

The case $k = 3$. Differential reduction

$$3\alpha\left(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)\right) = v_{xxx}.$$

A pair of reductions with different k – reduction to (1+1)

Reductions of interpolating system (i.e., the reduction with $k=1$, together with the reduction of some order $k \neq 1$ with a constant β).
For $k=2$ we obtain a system

$$\begin{aligned} &u_y + v_x u_x = (2\beta)^{-1} v_{xx}, \\ &v_x = \alpha u, \end{aligned}$$

which implies a hydrodynamic type equation (Hopf type equation) for u,

$$u_y + \alpha u u_x = \frac{\alpha}{2\beta} u_x.$$

The system for $k=3$ read

$$\begin{aligned} &\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x) = 3\beta^{-1} v_{xxx}, \\ &v_x = \alpha u, \end{aligned}$$

it implies an equation for u,

$$u_{yy} + \partial_x(2\alpha u_y u + \alpha^2 u_x u^2 + u u_x - \frac{\alpha}{3\beta} u_x) = 0,$$

which can be rewritten as a system of hydrodynamic type for two functions u, w,

$$
\begin{aligned}
w_y &= \left(\frac{\alpha}{3\beta} - \alpha^2 u^2 - u\right)u_x - 2\alpha u w_x, \\
u_y &= w_x.
\end{aligned}
$$

A system of equations of hydrodynamic type corresponding to the reduction of interpolating system of arbitrary order $k > 3$ can be written explicitly.

Two reductions of higher order

A simple example of a system defined by two reductions of higher order (reductions of the order 2 and 3),

$$
\begin{aligned}
&u_y + v_x u_x = (2\alpha)^{-1} v_{xx},\\
&\big(\partial_y(u_y + u_x v_x) + \partial_x(u_y v_x + u_x v_x^2 + u u_x)\big) = (3\beta)^{-1} v_{xxx}.
\end{aligned}
$$

A system of hydrodynamic type for the functions u, $w = v_x$,

$$
\begin{aligned}
u_y + w u_x &= (2\alpha)^{-1} w_x,\\
w_y &= \frac{2\alpha}{3\beta} w_x - w w_x - 2\alpha u u_x.
\end{aligned}
$$

The characterization of reductions in terms of the dressing data

A dressing scheme for the MS hierarchy

$$
\begin{aligned}
L_{\text{in}} &= F_1(L_{\text{out}}, M_{\text{out}}),\\
M_{\text{in}} &= F_2(L_{\text{out}}, M_{\text{out}}),
\end{aligned}
$$

$L_{\text{in}}(p,\mathbf{t})$, $M_{\text{in}}(p,\mathbf{t})$ are analytic inside the unit circle, the functions $L_{\text{out}}(p,\mathbf{t})$, $M_{\text{out}}(p,\mathbf{t})$ are analytic outside the unit circle with a prescribed singulariry defined by the series.
The Riemann problem implies the analyticity of the differential form

$$
\Omega_0 = \frac{\mathrm{d}L \wedge \mathrm{d}M}{\{L, M\}}
$$

and the generating relation for the hierarchy.

Let $G_1(\lambda,\mu)$, $G_2(\lambda,\mu)$ define an area-preserving diffeomorphism, $\mathbf{G} \in \mathrm{SDiff}(2)$,

$$\left|\frac{D(G_1,G_2)}{D(\lambda,\mu)}\right| = 1.$$

Let us fix a pair of analytic functions $f_1(\lambda,\mu)$, $f_2(\lambda,\mu)$ (the reduction data) and consider a problem

$$\begin{aligned} f_1(L_{\mathrm{in}},M_{\mathrm{in}}) &= G_1(f_1(L_{\mathrm{out}},M_{\mathrm{out}}), f_2(L_{\mathrm{out}},M_{\mathrm{out}})), \\ f_2(L_{\mathrm{in}},M_{\mathrm{in}}) &= G_2(f_1(L_{\mathrm{out}},M_{\mathrm{out}}), f_2(L_{\mathrm{out}},M_{\mathrm{out}})), \end{aligned}$$

which defines a reduction of the MS hierarchy. In terms of the Riemann problem for the MS hierarchy, which can be written in the form

$$(L_{\mathrm{in}},M_{\mathrm{in}}) = \mathbf{F}(L_{\mathrm{out}},M_{\mathrm{out}}),$$

the reduction condition for the dressing data reads

$$\mathbf{f}\circ\mathbf{F}\circ\mathbf{f}^{-1} \in \mathrm{SDiff}(2).$$

In terms of equations of the MS hierarchy the reduction is characterized by the condition

$$(\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_{\mathrm{out}} = (\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_{\mathrm{in}},$$

thus the differential form

$$\Omega_{\mathrm{red}} = \mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M)$$

is analytic in the complex plane, and reduced hierarchy is defined by the generating relation

$$(\mathrm{d}f_1(L,M)\wedge \mathrm{d}f_2(L,M))_- = 0.$$

Taking

$$\begin{aligned} f_1(L,M) &= L, \\ f_2(L,M) &= e^{-\alpha L^n} M, \end{aligned}$$

we obtain the generating relation

$$\left(e^{-\alpha L^k}\mathrm{d}L\wedge \mathrm{d}M\right)_- = 0,$$

coinciding with the generating relation for k-reduced MS hierarchy.

Thus we come to the following conclusion:

Proposition

In terms of the dressing data for the Riemann problem, the class of reductions (defined above) is characterized by the condition

$$\mathbf{f} \circ \mathbf{F} \circ \mathbf{f}^{-1} \in SDiff(2),$$

where the components of $\mathbf{f}$ *are defined as*

$$f_1(L, M) = L, \quad f_2(L, M) = e^{-\alpha L^n} M,$$

For the interpolating equation we have $f_1 = L$, $f_2 = e^{-\alpha L} M$, and the Riemann problem can be written in the form

$$\begin{aligned} L_{\text{in}} &= G_1(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}}), \\ M_{\text{in}} &= e^{\alpha G_1(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}})} G_2(L_{\text{out}}, e^{-\alpha L_{\text{out}}} M_{\text{out}}), \end{aligned}$$

where $\mathbf{G} \in \text{SDiff}(2)$.

Hamiltonian structure of interpolating reduction

Lax-Sato equations for the reduction with $k = 1$ (Dunajski interpolating equation) can be written in Hamiltonian form, but with the modified Poisson bracket (in collab. with S.V. Manakov). Indeed,

$$\{L, M\} = \exp \alpha(L - p) \Rightarrow e^{\alpha p}\{L, e^{-\alpha L} M\} = 1,$$

that indicates that the dynamics is Hamiltonian with the bracket $\{-, -\}' = e^{\alpha p}\{-, -\}$. The first flow of reduced hierarchy

$$\partial_y \mathbf{\Psi} = ((p - \alpha u)\partial_x - u_x \partial_p)\mathbf{\Psi},$$

can be written in Hamiltonian form

$$\begin{aligned} &\partial_y \mathbf{\Psi} = e^{\alpha p}\{H_1, \mathbf{\Psi}\}, \\ &H_1 = e^{-\alpha p}(u - \alpha^{-1}(p + \alpha^{-1})). \end{aligned}$$

It is possible to prove that all the flows of the reduced hierarchy are Hamiltonian with the bracket $\{-,-\}' = e^{\alpha p}\{-,-\}$, however, we don't have an explicit formula for H_n.
For higher reductions, there is an anti-symmetric invariant, but the corresponding 'bracket' doesn't satisfy the Jacobi identity.

Interpolating reduction for d2DTL case

Non-homogeneous linear equations

$$\partial_x \mathbf{\Phi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - (\phi_t \frac{m_x}{m_t} - \phi_x)\lambda\partial_\lambda \right) \mathbf{\Phi} + \beta\partial_t \frac{m_x}{m_t},$$

$$\partial_y \mathbf{\Phi} = \left(-\frac{1}{\lambda}\frac{e^{-\phi}}{m_t}\partial_t - \frac{1}{\lambda}\frac{(e^{-\phi})_t}{m_t}\lambda\partial_\lambda \right) \mathbf{\Phi} - \beta\frac{e^{-\phi}}{\lambda}\partial_t \frac{1}{m_t}, \qquad (1)$$

the substitution of solution $\ln\lambda$ to both equations gives the same reduction condition

$$\mathrm{e}^{\alpha\phi} = m_t, \quad \alpha = -\beta^{-1},$$

This reduction makes it possible to rewrite the d2DTL system as one equation for m,

$$m_{tt} = (m_t)^{\frac{1}{\alpha}}(m_{ty}m_x - m_{xy}m_t), \qquad (*)$$

or in the form of deformed d2DTL equation,

$$
\begin{aligned}
&(\mathrm{e}^{-\phi})_{tt} = m_t\phi_{xy} - m_x\phi_{ty}, \\
&m_t = \mathrm{e}^{\alpha\phi}.
\end{aligned}
$$

The limit $\alpha \to 0$ gives the d2DTL equation, the limit $\alpha \to \infty$ gives the equation introduced by Pavlov; Shabat and Martinez Alonso.
Equation $(*)$ is connested with the generalization of a dispersionless $(1+2)$-dimensional Harry Dym equation, Blaszak (2002), and also with an equation describing ASD vacuum metric with conformal symmetry, Dunajski and Tod (1999)

Hamiltonian structure of reduction, d2DTL case

(In collaboration with S.V.Manakov)

The Lax-Sato equations are Hamiltonian with the bracket

$$\{f, g\}' = \lambda^\alpha \{f, g\} = \lambda^{\alpha+1}(f_\lambda g_t - f_t g_\lambda).$$

$$\{\Lambda, M\} = \lambda^{-\alpha} \exp(\alpha\Lambda) \Rightarrow \{\Lambda, \exp(-\alpha\Lambda) M\}' = 1.$$

The Lax pair

$$\partial_x \mathbf{\Psi} = \left((\lambda + \frac{m_x}{m_t})\partial_t - \lambda(\phi_t \frac{m_x}{m_t} - \phi_x)\partial_\lambda \right) \mathbf{\Psi},$$

$$\partial_y \mathbf{\Psi} = \left(\frac{1}{\lambda} \frac{\mathrm{e}^{-\phi}}{m_t} \partial_t + \frac{(\mathrm{e}^{-\phi})_t}{m_t} \partial_\lambda \right) \mathbf{\Psi}$$

with the reduction $m_t = \mathrm{e}^{\alpha\phi}$ can be written in Hamiltonian form

$$\partial_x \mathbf{\Psi} = \{H_x, \mathbf{\Psi}\}', \quad H_x = (1-\alpha)^{-1}\lambda^{1-\alpha} - \alpha^{-1}\lambda^{-\alpha}\frac{m_x}{m_t},$$

$$\partial_y \mathbf{\Psi} = \{H_y, \mathbf{\Psi}\}', \quad H_y = -\frac{1}{\alpha+1}\lambda^{-\alpha-1} m_t^{-\frac{1}{\alpha}-1}.$$

THANK YOU!