Supercurrent in superconducting graphene

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Outline

- Electronic structure in the normal state
- Possible superconducting state
- BdG Dirac equations for SC graphene
- Problem of supercurrent
Electronic properties

Figure 1 (Color online) Graphene (top left) is a honeycomb lattice of carbon atoms. Graphite (top right) can be viewed a stack of graphene layers. Carbon nanotubes are rolled-up cylinders of graphene (bottom left). Fullerenes ($C_{60}$) are molecules consisting of wrapped graphene by the introduction of pentagons on the hexagonal lattice (Castro Neto et al., 2006a).

Electronic structure in the normal state

**Real space**

- $\delta_1$
- $\delta_2$
- $\delta_3$
- $a_1$
- $a_2$
- $K$
- $K'$
- $k_x$
- $k_y$
- $b_1$
- $b_2$

**(b) Brillouin zone**

$K + K' = b_1 + b_2$

$K = -K'$

Tight-binding Hamiltonian

$$H = -t \sum_{i,j,\sigma} \left[ \Psi_2^\dagger(\sigma, R_i) \Psi_1(\sigma, R_j) + \Psi_1^\dagger(\sigma, R_j) \Psi_2(\sigma, R_i) \right]$$

$$-t' \sum_{i,j,\sigma} \left[ \Psi_1^\dagger(\sigma, R_i) \Psi_1(\sigma, R_j) + \Psi_2^\dagger(\sigma, R_j) \Psi_2(\sigma, R_i) + h.c. \right]$$

$\Psi_2^\dagger(\sigma, R_i)$ creates a particle with spin $\sigma$ at a site $R_i$ of the sublattice 2

$\Psi_1(\sigma, R_j)$ annihilates a particle with spin $\sigma$ at a site $R_j$ of the sublattice 1.

The first sum runs over the nearest neighbor sites in different sublattices

$$R_j = R_i + \delta_n \ , \ n = 1, 2, 3$$

The second sum is over the next-nearest neighbors in the same sublattices.
Spectrum near the Dirac points

\[ |E - E_c| \approx \sqrt{3} \pi \gamma_0 \alpha |k - k_c| \quad (3.1) \]

Wallace (1947)

McClure (1957), Slonczewski and Weiss (1958)

Near the corner points

$\pm K$ in the Brillouin zone, $|k| \ll a^{-1}$

$$\Psi_{1}(R_{i}) = \frac{1}{\sqrt{N}} \sum_{k} \left[ e^{i(\mathbf{K}+\mathbf{k}) \cdot \mathbf{R}_{i}} \Psi_{1}(\mathbf{k}) + e^{i(-\mathbf{K}+\mathbf{k}) \cdot \mathbf{R}_{i}} \bar{\Psi}_{1}(\mathbf{k}) \right]$$

$$H = v_{F} \left[ \hat{\Psi}^{\dagger}(\mathbf{r})(\hat{\sigma} \cdot \hat{\mathbf{p}})\hat{\Psi}(\mathbf{r}) - \hat{\Psi}^{\dagger}(\mathbf{r})(\hat{\sigma}^{*} \cdot \hat{\mathbf{p}})\hat{\Psi}(\mathbf{r}) \right]$$

$H = v_{F} = 3a\ell/2$

$$\hat{\Psi} = \begin{pmatrix} \Psi_{1} \\ \Psi_{2} \end{pmatrix}, \quad \hat{\Psi}^{\dagger} = \begin{pmatrix} \Psi_{1}^{\dagger} & \Psi_{2}^{\dagger} \end{pmatrix}, \quad \hat{\mathbf{p}} = -i\hbar \nabla$$

Schrödinger equations

Near $\mathbf{K}$

$$v_{F}(\hat{\sigma} \cdot \hat{\mathbf{p}})\hat{\Psi}(\mathbf{r}) = E\hat{\Psi}(\mathbf{r})$$

$$-v_{F}(\hat{\sigma}^{*} \cdot \hat{\mathbf{p}})\hat{\Psi}(\mathbf{r}) = E\hat{\Psi}(\mathbf{r})$$

$\bar{\Psi}_{1} \rightarrow -\Psi_{2}$ and $\bar{\Psi}_{2} \rightarrow \Psi_{1}$.

$$E = \pm v_{F}p$$
Superconducting state

A hole excitation $\hat{\Psi}^h_K$ at $K \Rightarrow \hat{\Psi}^\dagger$ for the excitation at $-K$,

$$v_F(\sigma \cdot \hat{p})\hat{\Psi}^h_K(r) = E\hat{\Psi}^h_K(r)$$

Energy of particles and holes is measured from chemical potential $\mu$,

$$E = \mu \pm \epsilon$$

$$v_F(\sigma \cdot \hat{p})\hat{\Psi}^e_K(r) = (\mu + \epsilon)\hat{\Psi}^e_K(r)$$
$$v_F(\sigma \cdot \hat{p})\hat{\Psi}^h_K(r) = (\mu - \epsilon)\hat{\Psi}^h_K(r)$$

In the presence of magnetic field,

$$v_F\sigma \cdot \left(\hat{p} - \frac{e}{c}A\right)\hat{\Psi}^e_K(r) = (\mu + \epsilon)\hat{\Psi}^e_K(r)$$
$$v_F\sigma \cdot \left(\hat{p} + \frac{e}{c}A\right)\hat{\Psi}^h_K(r) = (\mu - \epsilon)\hat{\Psi}^h_K(r)$$

The Bogoliubov–de Gennes equations

$$v_F\hat{\sigma} \cdot \left(-i\nabla - \frac{e}{c}A\right)\hat{u} + \Delta\hat{v} = (E + \mu)\hat{u}$$
$$v_F\hat{\sigma} \cdot \left(i\nabla - \frac{e}{c}A\right)\hat{v} + \Delta^*\hat{u} = (E - \mu)\hat{v}$$

Uchoa, et al. (2005); Beenakker, Rev. Mod. Phys. (2008)
• Induced superconductivity


Fig. 1. (a) A scanning electron micrograph of sample A. (b) A schematic side view of the samples. The gray region indicates the graphene layers (thickness ~0.5–1 nm) in which the carrier concentration is expected to be modulated by the gate voltage.

• Intrinsic superconductivity

\[ \Delta = V \sum_k \left[ \langle \Psi_{1,\downarrow}^h (k) \Psi_{1,\uparrow}^e (k) \rangle + \langle \Psi_{2,\downarrow}^h (k) \Psi_{2,\uparrow}^e (k) \rangle \right] \]

Various mechanisms of pairing

- Phonon, Plasmon: *Uchoa, Castro Neto (2007)*
- RVB: *Black-Schaffer, Doniack (2007)*
- Phonons+edge states: *Sasaki et al (2007)*
- Hubbard model: *Zhao, Paramekanti (2006)*

Fig. 2. The zero-bias resistance of sample A as a function of temperature. The resistance at \( V_g = -35, 0 \text{ V} \) is indicated by filled symbols and that at \( V_g = 35, 70 \text{ V} \) is indicated by open symbols. The inset shows the gate-voltage dependence of the normal-state resistance. At \( V_g = V_g^p \approx 15 \text{ V} \), the normal-state resistance takes the maximum value.
**Normal-state spectrum**

(a) Undoped  
(b) Electron doped  
(c) Hole doped

\[ E_F = E_{F_0} + \mu. \]

**Electron spectrum**

\[ \xi_p = \pm \nu p - \mu \]

for spin states parallel and antiparallel to the momentum

\[ \hat{a}_\uparrow = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_x - ip_y} \\ p \\ \sqrt{p_x + ip_y} \end{pmatrix}, \quad \hat{a}_\downarrow = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{p_x - ip_y} \\ -p \\ \sqrt{p_x + ip_y} \end{pmatrix} \]
Model description of SC

\[ K \& Sonin, PRL (2008) \]

Current carrying state

\[ \Delta = |\Delta| e^{i k_s r}, \ k_s = \nabla \chi \]

\[ u(r) = u_p e^{i p_+ \cdot r/\hbar}, \ v(r) = v_p e^{i p_- \cdot r/\hbar}, \ p_\pm = p \pm \hbar k_s/2 \]

BdG equations

\[ \xi_p u_p + \Delta v_p = E_p u_p \]
\[ -\xi_p v_p + \Delta^* u_p = E_p v_p \]

For \( k_s \ll \xi_0^{-1} \sim \Delta_0/v \) within the first-order terms in \( v k_s \)

\[ u_p = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi_p}{E_p^{(0)}}}, \ v_p = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi_p}{E_p^{(0)}}} \]

\[ E_p = E_D + E_p^{(0)}, \ E_p^{(0)} = \sqrt{\xi_p^2 + |\Delta|^2}, \ \xi_p = \pm v_F p - \mu \]

Doppler energy

\[ E_D = \frac{d\xi_p}{dp} \frac{\hbar k_s}{2} = \pm \frac{\hbar v_F (p \cdot k_s)}{2p} \]
Current

\[ j = 2e \sum_p \left[ \frac{\partial \xi_{p+}}{\partial p} |u_p|^2 n(E_p) - \frac{\partial \xi_{p-}}{\partial p} |v_p|^2 [1 - n(E_p)] \right] \]

Linear response for small \( E_D \ll \Delta_0 \)

\[ j = e \int \frac{d^2 p}{4\pi^2 \hbar} \frac{\partial \xi_p}{\partial p} \left( \frac{\partial \xi_p}{\partial p} \cdot k_s \right) \frac{\partial}{\partial \xi_p} \left[ \frac{\xi_p}{2E_p^{(0)}} [1 - 2n(E_p^{(0)})] \right] \]

\[ + 2e \int \frac{d^2 p}{4\pi^2 \hbar^2} \frac{\partial \xi_p}{\partial p} \left[ n(E_p) - n(E_p^{(0)}) \right] . \]

2D density

\[ j = \frac{e |\Delta| k_s}{4\pi \hbar} \Lambda \left( \frac{\mu}{|\Delta|}, \frac{T}{|\Delta|} \right) \]

- Zero temperature

\[ \Lambda(x, 0) = 2 + \frac{x^2}{\sqrt{x^2 + 1^2}} - \frac{1}{\sqrt{x^2 + 1}} \]

- Zero doping

\[ \Lambda \left( 0, \frac{T}{|\Delta|} \right) = \tanh \frac{|\Delta|}{2T} = \begin{cases} |\Delta|/2T_c, & T \rightarrow T_c \\ 1, & T \rightarrow 0 \end{cases} \]

Current is finite at \( T=0 \)

As distinct from: Uchoa, Cabrera, & Castro Neto (2005)
\[ j = QA \]

\[ A \to A - \frac{hc}{2e} k \]

\[ Q \Rightarrow [Q_{\perp}(\Delta) - Q_{\perp}(0)] \]

\[ Q_{\perp}(\Delta_s) - Q_{\perp}(0) \xrightarrow{T \to 0} - \frac{|\mu| e^2 v_F}{d \pi v_{\Delta c}} \]

Current vanishes for \( \mu \to 0 \)

FIG. 16. London kernel dependence with temperature in the cone approximation \( (g'/g_c = 1.1) \). Plots for \( 0 \leq |\mu|/\alpha \leq 0.16 \), from the bottom to the top, in fixed intervals of 0.02. \( Q(0) - Q(\Delta) \) in units of \( e^2 v_F \alpha/(2 \pi d v_{\Delta c}) \). (In our case \( \alpha = \xi_m, v_{\Delta} = v_F \))
Supercurrent at $T \ll \Delta$

$$j = ev^2 \sum_p n(n \cdot k_s) \frac{\partial}{\partial \xi_p} \left[ \frac{\xi_p}{2E_p^{(0)}} \right]$$

Usual 3D case

$$j = \frac{e\nu_F v^2 k_s}{3} \int_0^\infty d\xi \frac{|\Delta|^2}{(\xi^2 + |\Delta|^2)^{3/2}} = \frac{Nek_s}{2m} = Nev_s$$

Dirac point, zero doping

$$j = \frac{ek_s}{4\pi} \int_0^\infty \xi d\xi \frac{|\Delta|^2}{(\xi^2 + |\Delta|^2)^{3/2}} = \frac{e|\Delta|k_s}{4\pi}$$

Total number of electrons

$$\xi \sim |\Delta|$$

Supercurrent is finite despite the zero DOS at $\xi \to 0$
Microscopic description of the current-carrying state

\[ \hat{u}_p = \hat{u} e^{i(p+k/2) \cdot r}, \quad \hat{v}_p = \hat{v} e^{i(p-k/2) \cdot r}, \quad \Delta = |\Delta| e^{iK \cdot r}, \]

**BdG equations**

\[ v_F \hat{\sigma} \cdot (p + k/2) \hat{u} + \Delta \hat{v} = (E + \mu) \hat{u}, \]
\[ -v_F \hat{\sigma} \cdot (p - k/2) \hat{v} + \Delta^* \hat{u} = (E - \mu) \hat{v}. \]

**Supercurrent**

\[ j = 2ev_F \sum_{p,\alpha} \left[ \hat{u}_{p,\alpha}^\dagger \hat{\sigma} \hat{u}_{p,\alpha} f_{p,\alpha} - \hat{v}_{p,\alpha}^\dagger \hat{\sigma} \hat{v}_{\alpha} (1 - f_{p,\alpha}) \right]. \]

\[ j = -e v_F \sum_{p,\alpha} \left[ \hat{u}_{p,\alpha}^\dagger \hat{\sigma} \hat{u}_{p,\alpha} + \hat{v}_{p,\alpha}^\dagger \hat{\sigma} \hat{v}_{\alpha} \right] (1 - 2f_{p,\alpha}) \]

\[ j = \int \frac{d^2 p}{(2\pi)^2} [j_K(p) + j_{-K}(p)] \]

\[ j_K(p) = -e v_F \sum_{\alpha=1}^{4} \hat{u}_{p,\alpha}^\dagger \hat{\sigma} \hat{u}_{p,\alpha} [1 - 2f_{p,\alpha}] \]

\[ j_{-K}(p) = -e v_F \sum_{\alpha=1}^{4} \hat{v}_{p,\alpha}^\dagger \hat{\sigma} \hat{v}_{p,\alpha} [1 - 2f_{p,\alpha}] \]
Zero-current ground state

Define spinors that satisfy

$$(\hat{\sigma} \cdot \mathbf{p})\hat{a}_{\uparrow,\downarrow} = \pm p \hat{a}_{\uparrow,\downarrow}$$

The states with pseudospin parallel and anti-parallel to the momentum

$$\hat{a}_{\uparrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x-ip_y}{p}} \\ \sqrt{\frac{p_x+ip_y}{p}} \end{pmatrix}, \quad \hat{a}_{\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x-ip_y}{p}} \\ -\sqrt{\frac{p_x+ip_y}{p}} \end{pmatrix}$$

The spinors $\hat{a}_{\uparrow}$ and $\hat{a}_{\downarrow}$ are eigenstates of excitations in the normal graphene.

We introduce vectors in the Nambu space,

$$\tilde{\psi} = \begin{pmatrix} \hat{u} \\ \hat{\psi} \end{pmatrix}, \quad \tilde{\psi}^+ = (\hat{u}^\dagger, \hat{\psi}^\dagger), \quad \tilde{\psi}_\alpha^+ \tilde{\psi}_\beta = \delta_{\alpha\beta}$$
Eigen-states for zero current

For $\uparrow$ spin

$$E^{(0)}_{1,2} = \pm E_\uparrow, \quad E_\uparrow = \sqrt{(v_{FP} - \mu)^2 + |\Delta|^2}$$

$$\begin{pmatrix}
\hat{u}^{(0)}_1 \\
\hat{v}^{(0)}_1
\end{pmatrix} = \begin{pmatrix}
u_\uparrow \\
-v_\uparrow
\end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p} \cdot \mathbf{r}}, \quad \begin{pmatrix}
\hat{u}^{(0)}_2 \\
\hat{v}^{(0)}_2
\end{pmatrix} = \begin{pmatrix}v_\uparrow \\
-u_\uparrow
\end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p} \cdot \mathbf{r}}$$

For $\downarrow$ spin

$$E^{(0)}_{3,4} = \pm E_\downarrow, \quad E_\downarrow = \sqrt{(v_{FP} + \mu)^2 + |\Delta|^2}$$

$$\begin{pmatrix}
\hat{u}^{(0)}_3 \\
\hat{v}^{(0)}_3
\end{pmatrix} = \begin{pmatrix}u_\downarrow \\
v_\downarrow
\end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p} \cdot \mathbf{r}}, \quad \begin{pmatrix}
\hat{u}^{(0)}_4 \\
\hat{v}^{(0)}_4
\end{pmatrix} = \begin{pmatrix}v_\downarrow \\
-u_\downarrow
\end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p} \cdot \mathbf{r}}.$$

$$u_\uparrow = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{v_{FP} - \mu}{E_\uparrow}}, \quad v_\uparrow = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{v_{FP} - \mu}{E_\uparrow}},$$

$$u_\downarrow = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{v_{FP} + \mu}{E_\downarrow}}, \quad v_\downarrow = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{v_{FP} + \mu}{E_\downarrow}}.$$
Current-carrying state

Spectrum

\[
(E^2 - \mu^2)^2 - 2|\Delta|^2(E^2 - \mu^2) + |\Delta|^4 + 2|\Delta|^2v_F^2p_+p_- \\
-(E + \mu)^2v_F^2p_-^2 - (E - \mu)^2v_F^2p_+^2 + v_F^4p_+^2p_-^2 = 0
\]

where \( p_\pm = p \pm k/2 \).

Two limiting cases

- \( v_Fk \ll \mu \)

\[
E_\alpha = E^{(0)}_\alpha + E^{(1)}_\alpha \\
E^{(1)}_{1,2} = -E^{(1)}_{3,4} = E_D \\ E_D = v_F(p \cdot k)/2p
\]

Doppler-shifted energies

- \( \mu = 0 \):

\[
E^2_\pm = |\Delta|^2 + v_F^2(p^2 + k^2/4) \pm \sqrt{|\Delta|^4v_F^4k^2 + v_F^4(p \cdot k)^2}
\]

No Doppler shift

Degenerate state, \( E^2_+ = E^2_- \) for \( k = 0 \)
Correction to the supercurrent diverges because it extends over the entire BZ.
Regularization of the divergence

The current-carrying state

\[ \hat{u}_p = \hat{u} e^{i(p+k/2) \cdot r} , \quad \hat{v}_p = \hat{v} e^{i(p-k/2) \cdot r} , \]

contains contributions from the overall momentum shift in the BZ.

Since \( v_K(p) = u_{-K}(-p) \), a homogeneous shift by \( k \) gives \( p' \rightarrow p' + k \) and

\[ u_K(p) \rightarrow u_K(p + k) , \quad u_{-K}(-p) \rightarrow u_{-K}(-p + k) \]

Therefore,

\[ u_K(p) \rightarrow u_K(p + k) , \quad v_K(p) \rightarrow v_K(p - k) \]
The same result is obtained if one subtracts the normal current

\[ j^{(0)} = \int \frac{d^2p'}{(2\pi)^2} \left[ j^{(0)}_{K}(p' - k/2) + j^{(0)}_{-K}(p' + k/2) \right] \]

\[ = \int \frac{d^2p}{(2\pi)^2} \left[ j^{(0)}_{K}(p) + j^{(0)}_{-K}(p) - \left( k \cdot \frac{\partial}{\partial p} \right) j^{(0)}_{K}(p) \right]. \]

Here \( j^{(0)}_{K}(p) + j^{(0)}_{-K}(p) = 0 \). As a result

\[ j^{(0)} = -\int \frac{d\phi}{(2\pi)^2} \left[ (p \cdot k) j^{(0)}_{K}(p) \right]_{\phi \to \Delta, T}. \]

\[ j^{(0)}_{K}(p) = -ev_{F} \left[ \frac{v_{Fp} - \mu}{E_{\uparrow}/2T} + \frac{v_{Fp} + \mu}{E_{\downarrow}/2T} \tanh \frac{E_{\downarrow}}{2T} \right] \frac{p}{p} \]

\[ j^{(0)}_{K}(p) \bigg|_{p, \infty} = -2ev_{F} \left[ 1 + \frac{\Delta^2}{2v_{Fp}^2} \right] \frac{p}{p} \rightarrow -2ev_{F} \frac{p}{p} \]

and

\[ (p \cdot k) j^{(0)}_{K}(p) \bigg|_{p \to \infty} \rightarrow -2ev_{F} \frac{p}{p}(p \cdot k) \]

The same result is obtained if one subtracts the normal current
For $T = 0$ the supercurrent becomes

$$j = \frac{ek}{2\pi} \left[ \sqrt{\mu^2 + |\Delta|^2} + \frac{|\Delta|^2}{|\mu|} \ln \left( \frac{|\mu| + \sqrt{\mu^2 + |\Delta|^2}}{|\Delta|} \right) \right]$$

- For $\mu \gg |\Delta|$
  $$j = e|\mu|k/2\pi$$

- For $\mu \ll |\Delta|$
  $$j = e|\Delta|k/\pi$$

This result formally holds within the linear approximation which assumes $v_Fk \ll \mu$. Therefore, one has to put $k \to 0$ first and then assume $\mu \ll |\Delta|$. 

![Graph showing the relationship between $\Phi$ and $\mu/\Delta$]
What if $\mu \ll v_F k$?

The spectrum for $\mu = 0$:

$$E_{\pm}^2 = |\Delta|^2 + v_F^2 (p^2 + k^2/4) \pm \sqrt{|\Delta|^2 v_F^2 k^2 + v_F^4 (p \cdot k)^2}$$

No Doppler energy

The zero-current state is degenerate:

$$E_+ = E_- = E_0 = \sqrt{v_F^2 p^2 + |\Delta|^2}$$

$$E_1 - E_3 = E_0, \quad E_2 - E_4 = -E_0$$

Requires a special consideration

One can show that the linear-response results $v_F k \ll |\Delta|$ holds irrespectively of the relation between $v_F k$ and $\mu$. 
Conclusions

- No qualitative difference between the critical temperature, superconducting gap and supercurrent obtained for the simple model and for the two-valley BdG-Dirac description

  - Slightly different parametric dependence of supercurrent
  
  - The $\mu \gg |\Delta|$ result gives 2 times larger current than in the simple model due to two times larger number of cones.

  - However, the $\mu \ll |\Delta|$ limit gives 4 times larger current. This is due to more subtle differences originating from interference of four ground states.

- The supercurrent is finite at any doping level as long as superconductivity exists