Supercurrent in superconducting graphene

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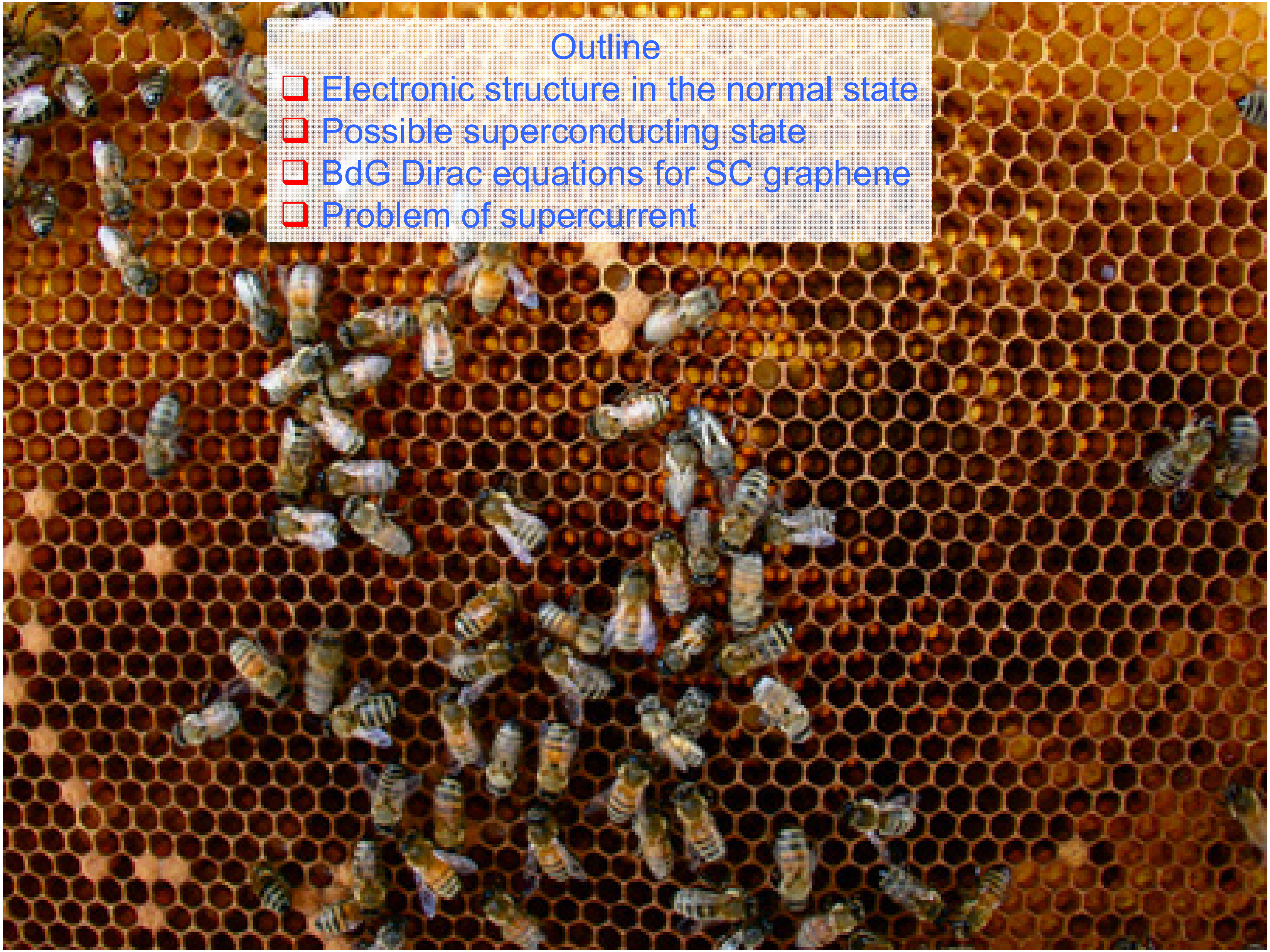


Outline

- Electronic structure in the normal state
- Possible superconducting state
- BdG Dirac equations for SC graphene
- Problem of supercurrent

Electronic properties

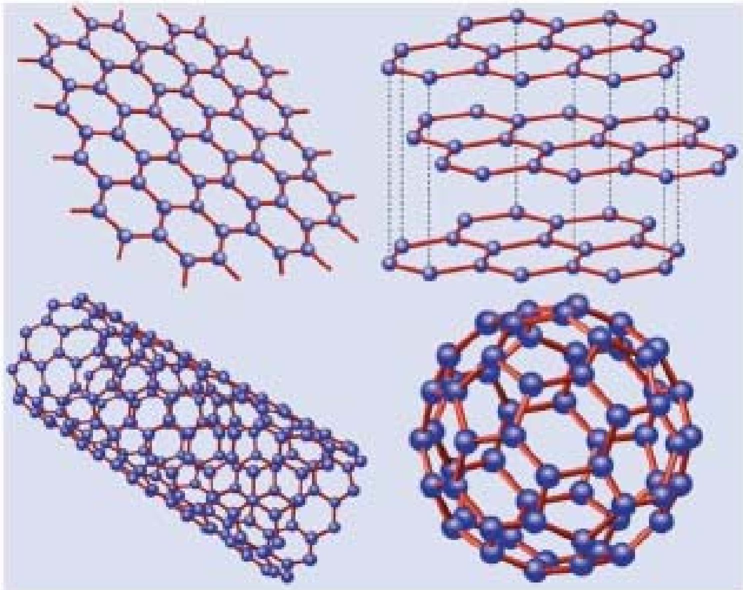
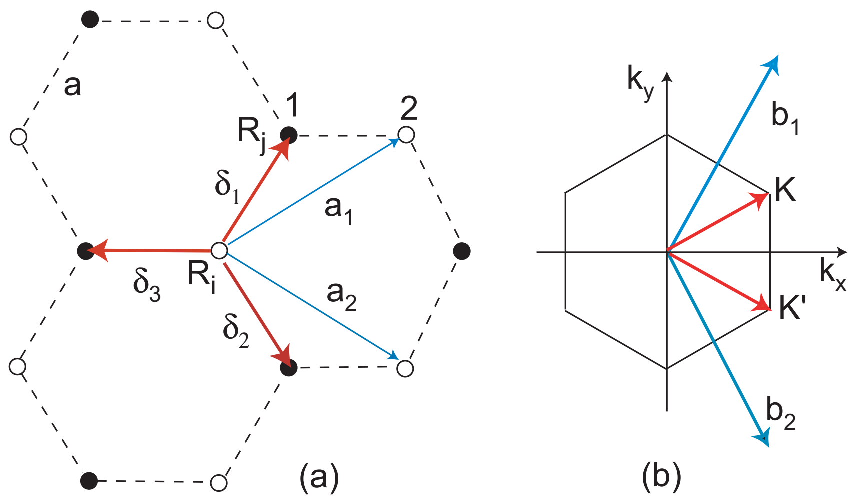


Figure 1 (Color online) Graphene (top left) is a honeycomb lattice of carbon atoms. Graphite (top right) can be viewed a stack of graphene layers. Carbon nanotubes are rolled-up cylinders of graphene (bottom left). Fullerenes (C_{60}) are molecules consisting of wrapped graphene by the introduction of pentagons on the hexagonal lattice (Castro Neto *et al.*, 2006a).

Normal properties: *Novoselov et al., Nature (2005)*

Electronic structure in the normal state

Real space



Brillouin zone

$$\mathbf{K} + \mathbf{K}' = \mathbf{b}_1 + \mathbf{b}_2$$

$$\mathbf{K} = -\mathbf{K}'$$

Tight-binding Hamiltonian

$$H = -t \sum_{i,j,\sigma} \left[\Psi_2^\dagger(\sigma, \mathbf{R}_i)\Psi_1(\sigma, \mathbf{R}_j) + \Psi_1^\dagger(\sigma, \mathbf{R}_j)\Psi_2(\sigma, \mathbf{R}_i) \right]$$

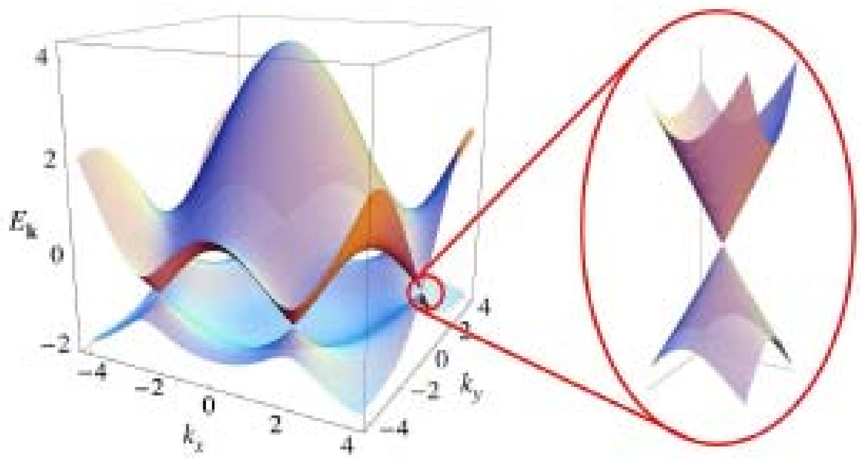
$$-t' \sum_{i,j,\sigma} \left[\Psi_1^\dagger(\sigma, \mathbf{R}_i)\Psi_1(\sigma, \mathbf{R}_j) + \Psi_2^\dagger(\sigma, \mathbf{R}_j)\Psi_2(\sigma, \mathbf{R}_i) + h.c. \right]$$

$\Psi_2^\dagger(\sigma, \mathbf{R}_i)$ creates a particle with spin σ at a site $\mathbf{R}_i$ of the sublattice 2
$\Psi_1(\sigma, \mathbf{R}_j)$ annihilates a particle with spin σ at a site $\mathbf{R}_j$ of the sublattice 1.
The first sum runs over the nearest neighbor sites in different sublattices

$$\mathbf{R}_j = \mathbf{R}_i + \boldsymbol{\delta}_n \ , \ n = 1, 2, 3$$

The second sum is over the next-nearest neighbors in the same sublattices.

Spectrum near the Dirac points



$$|E - E_c| \approx \sqrt{3}\pi\gamma_0 a|\mathbf{k} - \mathbf{k}_c| \qquad (3.1)$$

Wallace (1947)

McClure (1957),
Slonczewski and Weiss (1958)

Review: *Castro Neto et al*., Rev. Mod. Phys. v.81,109 (2009); arXiv:0709.1163

Near the corner points

$$\pm\mathbf{K} \text{ in the Brillouin zone, } |\mathbf{k}| \ll a^{-1}$$

$$\Psi_1(\mathbf{R}_i) = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} \left[e^{i(\mathbf{K}+\mathbf{k})\cdot\mathbf{R}_i} \Psi_1(\mathbf{k}) + e^{i(-\mathbf{K}+\mathbf{k})\cdot\mathbf{R}_i} \bar{\Psi}_1(\mathbf{k}) \right]$$

$$H = v_F \left[\hat{\Psi}^\dagger(\mathbf{r})(\hat{\boldsymbol{\sigma}} \cdot \check{\mathbf{p}})\hat{\Psi}(\mathbf{r}) - \hat{\bar{\Psi}}^\dagger(\mathbf{r})(\hat{\boldsymbol{\sigma}}^* \cdot \check{\mathbf{p}})\hat{\bar{\Psi}}(\mathbf{r}) \right] \ , \ v_F = 3at/2$$

$$\hat{\Psi} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \ , \ \hat{\Psi}^\dagger = \begin{pmatrix} \Psi_1^\dagger \ , \ \Psi_2^\dagger \end{pmatrix} \ , \ \check{\mathbf{p}} = -i\hbar\boldsymbol{\nabla}$$

Schrödinger equations

Near $\mathbf{K}$

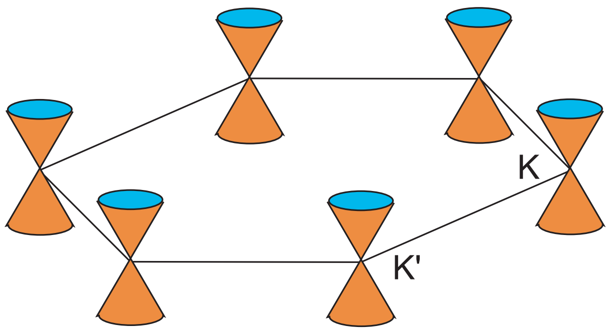
$$v_F(\hat{\boldsymbol{\sigma}} \cdot \check{\mathbf{p}})\hat{\Psi}(\mathbf{r}) = E\hat{\Psi}(\mathbf{r})$$

Near $-\mathbf{K}$

$$-v_F(\hat{\boldsymbol{\sigma}}^* \cdot \check{\mathbf{p}})\hat{\bar{\Psi}}(\mathbf{r}) = E\hat{\bar{\Psi}}(\mathbf{r})$$

$$\bar{\Psi}_1 \to -\Psi_2 \text{ and } \bar{\Psi}_2 \to \Psi_1.$$

$$E = \pm v_F p$$



Superconducting state

A hole excitation $\hat{\Psi}^h_{\mathbf{K}}$ at $\mathbf{K}$ $\Rightarrow$ $\hat{\bar{\Psi}}^\dagger$ for the excitation at $-\mathbf{K}$,

$$v_F(\boldsymbol{\sigma}\cdot\check{\mathbf{p}})\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r}) = E\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r})$$

Energy of particles and holes is measured from chemical potential μ,

$$E = \mu \pm \epsilon$$

$$v_F(\boldsymbol{\sigma}\cdot\check{\mathbf{p}})\hat{\Psi}^e_{\mathbf{K}}(\mathbf{r}) = (\mu+\epsilon)\hat{\Psi}^e_{\mathbf{K}}(\mathbf{r})$$
$$v_F(\boldsymbol{\sigma}\cdot\check{\mathbf{p}})\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r}) = (\mu-\epsilon)\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r})$$

In the presence of magnetic field,

$$v_F\boldsymbol{\sigma}\cdot\left(\check{\mathbf{p}}-\frac{e}{c}\mathbf{A}\right)\hat{\Psi}^e_{\mathbf{K}}(\mathbf{r}) = (\mu+\epsilon)\hat{\Psi}^e_{\mathbf{K}}(\mathbf{r})$$
$$v_F\boldsymbol{\sigma}\cdot\left(\check{\mathbf{p}}+\frac{e}{c}\mathbf{A}\right)\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r}) = (\mu-\epsilon)\hat{\Psi}^h_{\mathbf{K}}(\mathbf{r})$$

The Bogoliubov–de Gennes equations

$$v_F\hat{\boldsymbol{\sigma}}\cdot\left(-i\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)\hat{u}+\Delta\hat{v} = (E+\mu)\hat{u}$$
$$v_F\hat{\boldsymbol{\sigma}}\cdot\left(i\boldsymbol{\nabla}-\frac{e}{c}\mathbf{A}\right)\hat{v}+\Delta^*\hat{u} = (E-\mu)\hat{v}$$

Uchoa, et al. (2005); Beenakker, Rev. Mod. Phys. (2008)

• Induced superconductivity

Sato et al. Physica E (2008)

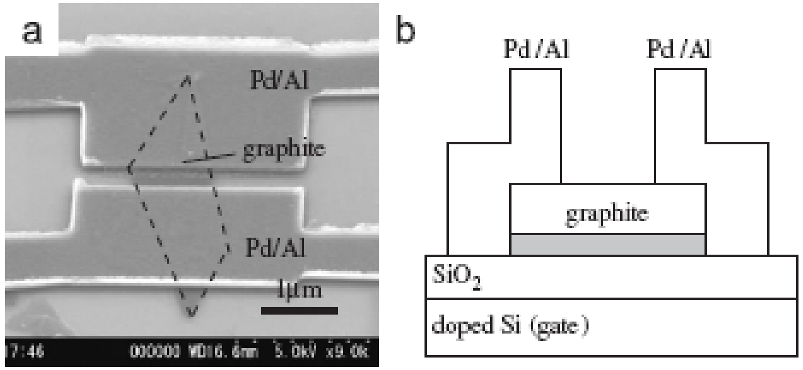


Fig. 1. (a) A scanning electron micrograph of sample A. (b) A schematic side view of the samples. The gray region indicates the graphene layers (thickness ~0.5−1 nm) in which the carrier concentration is expected to be modulated by the gate voltage.

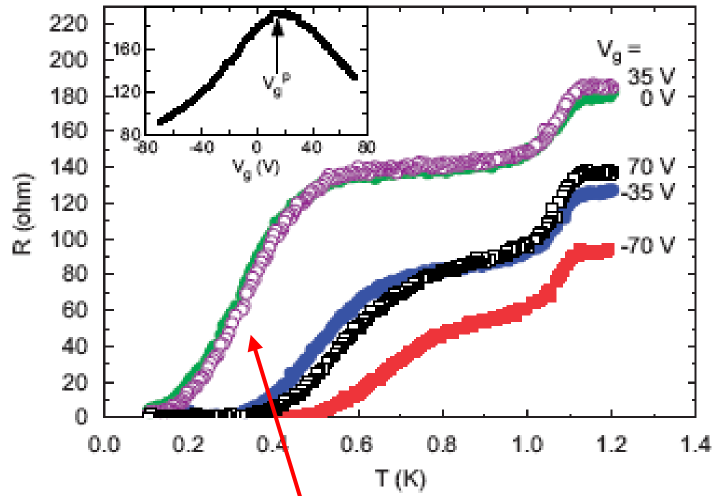


Fig. 2. The zero-bias resistance of sample A as a function of temperature. The resistance at $V_g = -70, -35$, 0 V is indicated by filled symbols and that at $V_g = 35$, 70 V is indicated by open symbols. The inset shows the gate-voltage dependence of the normal-state resistance. At $V_g = V_g^p \approx 15$ V, the normal-state resistance takes the maximum value.

Induced SC transition in graphene

• Intrinsic superconductivity

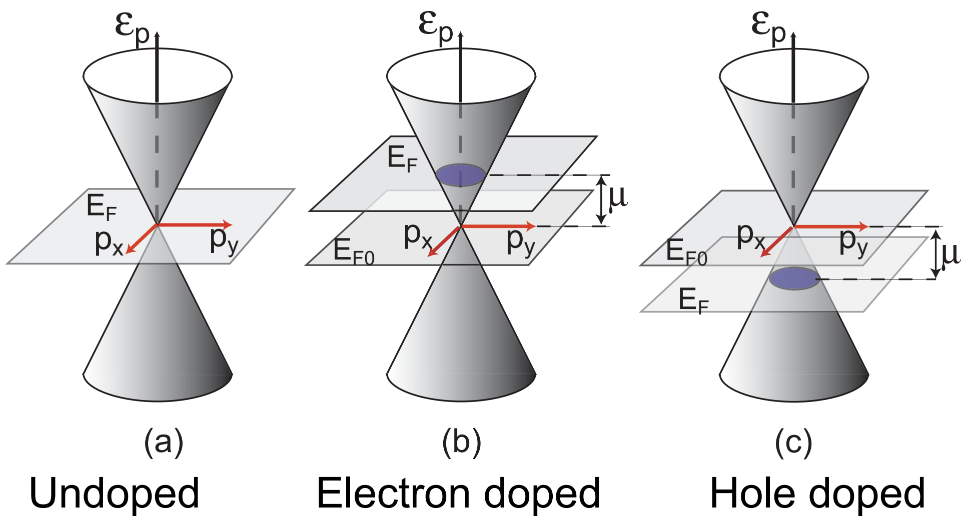
Order parameter

$$\Delta = V{\sum_{\mathbf{k}}}' \left[\left\langle \Psi^{h\dagger}_{1,\downarrow}(\mathbf{k}) \Psi^{e}_{1,\uparrow}(\mathbf{k}) \right\rangle + \left\langle \Psi^{h\dagger}_{2,\downarrow}(\mathbf{k}) \Psi^{e}_{2,\uparrow}(\mathbf{k}) \right\rangle \right]$$

Various mechanisms of pairing

Phonon, Plasmon:	*Uchoa, Castro Neto (2007),*
RVB:	*Black-Schaffer, Doniack (2007),*
Phonons+edge states:	*Sasaki et al (2007)*
Hubbard model:	*Zhao, Paramekanti (2006)*

Normal-state spectrum



$$E_F = E_{F0} + \mu.$$

Electron spectrum

$$\xi_{\mathbf{p}} = \pm vp - \mu$$

for spin states parallel and aniparallel to the momentum

$$\hat{a}_{\uparrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - ip_y}{p}} \\ \sqrt{\frac{p_x + ip_y}{p}} \end{pmatrix}, \quad \hat{a}_{\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - ip_y}{p}} \\ -\sqrt{\frac{p_x + ip_y}{p}} \end{pmatrix}$$

Model description of SC

K & Sonin, PRL (2008)

Current carrying state

$$\Delta = |\Delta| e^{i\mathbf{k}_s \mathbf{r}} \ , \ \mathbf{k}_s = \nabla \chi$$

$$u(\mathbf{r}) = u_{\mathbf{p}} e^{i\mathbf{p}_+ \cdot \mathbf{r}/\hbar} \ , \ v(\mathbf{r}) = v_{\mathbf{p}} e^{i\mathbf{p}_- \cdot \mathbf{r}/\hbar} \ , \ \mathbf{p}_\pm = \mathbf{p} \pm \hbar \mathbf{k}_s/2$$

BdG equations

$$\xi_{\mathbf{p}} u_{\mathbf{p}} + \Delta v_{\mathbf{p}} = E_{\mathbf{p}} u_{\mathbf{p}}$$
$$-\xi_{\mathbf{p}} v_{\mathbf{p}} + \Delta^* u_{\mathbf{p}} = E_{\mathbf{p}} v_{\mathbf{p}}$$

For $k_s \ll \xi_0^{-1} \sim \Delta_0/v$ within the first-order terms in vk_s

$$u_{\mathbf{p}} = \frac{1}{\sqrt{2}} \sqrt{1 + \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}^{(0)}}} \ , \ v_{\mathbf{p}} = \frac{1}{\sqrt{2}} \sqrt{1 - \frac{\xi_{\mathbf{p}}}{E_{\mathbf{p}}^{(0)}}}$$

$$E_{\mathbf{p}} = E_D + E_{\mathbf{p}}^{(0)} \ , \ E_{\mathbf{p}}^{(0)} = \sqrt{\xi_{\mathbf{p}}^2 + |\Delta|^2} \ , \ \xi_{\mathbf{p}} = \pm v_F p - \mu$$

Doppler energy

$$E_D = \frac{d\xi_p}{d\mathbf{p}} \frac{\hbar \mathbf{k}_s}{2} = \pm \frac{\hbar v_F (\mathbf{p} \cdot \mathbf{k}_s)}{2p}$$

Current

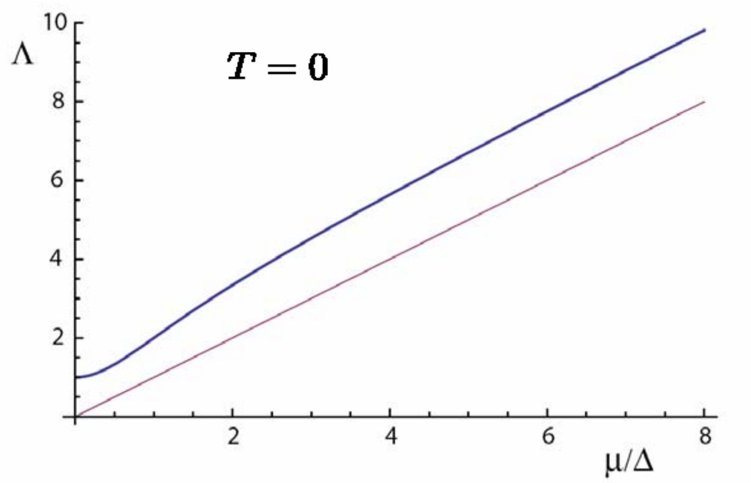
$$\mathbf{j} = 2e\sum_{\mathbf{p}}\left[\frac{\partial\xi_{\mathbf{p}+}}{\partial\mathbf{p}}|u_{\mathbf{p}}|^2 n(E_{\mathbf{p}}) - \frac{\partial\xi_{\mathbf{p}-}}{\partial\mathbf{p}}|v_{\mathbf{p}}|^2[1-n(E_{\mathbf{p}})]\right]$$

Linear response for small $E_D \ll \Delta_0$

$$\mathbf{j} = e\int\frac{d^2p}{4\pi^2\hbar}\frac{\partial\xi_{\mathbf{p}}}{\partial\mathbf{p}}\left(\frac{\partial\xi_{\mathbf{p}}}{\partial\mathbf{p}}\cdot\mathbf{k}_s\right)\frac{\partial}{\partial\xi_{\mathbf{p}}}\left[\frac{\xi_{\mathbf{p}}}{2E^{(0)}_{\mathbf{p}}}[1-2n(E^{(0)}_{\mathbf{p}})]\right]$$
$$+2e\int\frac{d^2p}{4\pi^2\hbar^2}\frac{\partial\xi_{\mathbf{p}}}{\partial\mathbf{p}}\left[n(E_{\mathbf{p}})-n(E^{(0)}_{\mathbf{p}})\right] \ .$$

2D density

$$\mathbf{j} = \frac{e|\Delta|\mathbf{k}_s}{4\pi\hbar}\Lambda\left(\frac{\mu}{|\Delta|},\frac{T}{|\Delta|}\right)$$



- Zero temperature

$$\Lambda(x,0) = 2+\frac{x^2}{\sqrt{x^2+1^2}}-\frac{1}{\sqrt{x^2+1}}$$

- Zero doping

$$\Lambda\left(0,\frac{T}{|\Delta|}\right) = \tanh\frac{|\Delta|}{2T} = \begin{cases}|\Delta|/2T_c\ , & T\to T_c\\ 1\ , & T\to 0\end{cases}$$

Current is finite at T=0

As distinct from: *Uchoa, Cabrera, & Castro Neto (2005)*

Uchoa, Cabrera, & Castro Neto (PRB, 2005)

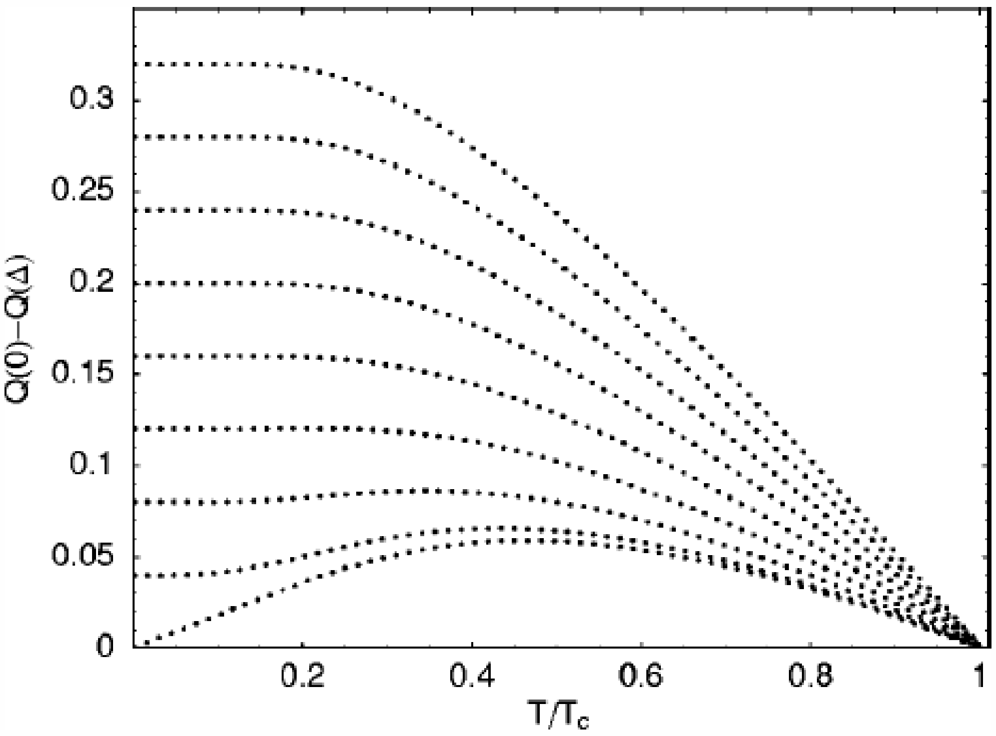


FIG. 16. London kernel dependence with temperature in the cone approximation (g/g_c=1.1). Plots for $0 \leqslant |\mu|/\alpha \leqslant 0.16$, from the bottom to the top, in fixed intervals of 0.02. $Q(0)-Q(\Delta_s)$ in units of $e^2 v_F \alpha/(2\pi d v_\Delta c)$. (In our case $\alpha \equiv \xi_m$, $v_\Delta = v_F$)

$$j = QA$$

$$\mathbf{A} \to \mathbf{A} - \frac{\hbar c}{2e}\mathbf{k}$$

$$\boldsymbol{Q \Rightarrow [Q_\perp(\Delta) - Q_\perp(0)]}$$

$$Q_\perp(\Delta_s) - Q_\perp(0) \xrightarrow{T \to 0} -\frac{|\mu|}{d}\frac{e^2 v_F}{\pi v_\Delta c}$$

Current vanishes for μ→0

Supercurrent at T<<Δ

$$\mathbf{j} = ev^2 \sum_{\mathbf{p}} \mathbf{n}\,(\mathbf{n}\cdot\mathbf{k}_s)\,\frac{\partial}{\partial \xi_{\mathbf{p}}}\left[\frac{\xi_{\mathbf{p}}}{2E^{(0)}_{\mathbf{p}}}\right]$$

Usual 3D case

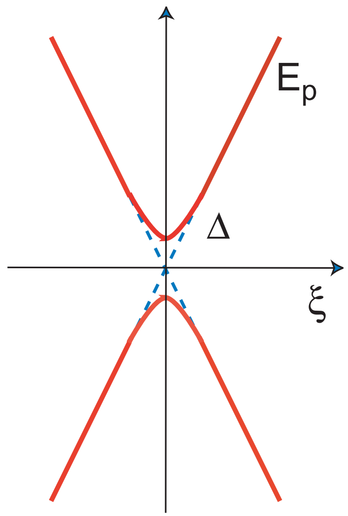
Total number of electrons

$$\mathbf{j} = \frac{e\nu_F v^2 \mathbf{k}_s}{3}\int_0^\infty d\xi\,\frac{|\Delta|^2}{(\xi^2+|\Delta|^2)^{3/2}} = \frac{Ne\mathbf{k}_s}{2m} = Ne\mathbf{v}_s$$

$\xi \sim |\Delta|$

Dirac point, zero doping

$$\mathbf{j} = \frac{e\mathbf{k}_s}{4\pi}\int_0^\infty \xi\, d\xi\,\frac{|\Delta|^2}{(\xi^2+|\Delta|^2)^{3/2}} = \frac{e|\Delta|\mathbf{k}_s}{4\pi}$$



Supercurrent is finite despite the zero DOS at $\xi \to 0$

Microscopic description of the current-carrying state

$$\hat{u}_{\mathbf{p}} = \hat{u} e^{i(\mathbf{p}+\mathbf{k}/2)\cdot\mathbf{r}} \ , \ \hat{v}_{\mathbf{p}} = \hat{v} e^{i(\mathbf{p}-\mathbf{k}/2)\cdot\mathbf{r}} \ , \ \Delta = |\Delta| e^{i\mathbf{k}\cdot\mathbf{r}}$$

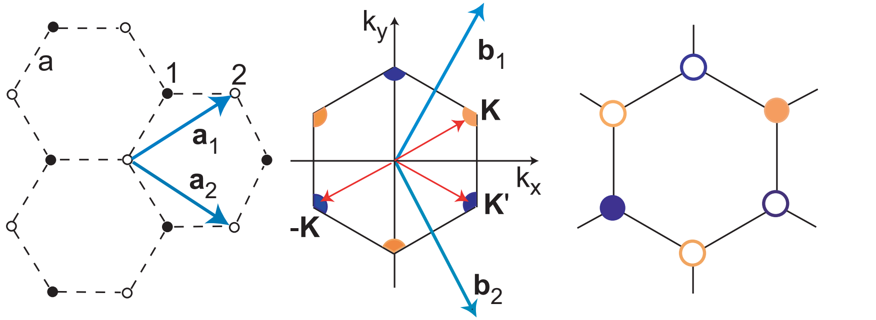
BdG equations

$$v_F \hat{\boldsymbol{\sigma}} \cdot (\mathbf{p} + \mathbf{k}/2)\hat{u} + \Delta \hat{v} = (E+\mu)\hat{u} \ ,$$
$$-v_F \hat{\boldsymbol{\sigma}} \cdot (\mathbf{p} - \mathbf{k}/2)\hat{v} + \Delta^* \hat{u} = (E-\mu)\hat{v} \ .$$

Supercurrent

$$\mathbf{j} = 2ev_F \sum_{\mathbf{p},\alpha} \left[\hat{u}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p},\alpha} f_{\mathbf{p},\alpha} - \hat{v}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{v}_\alpha (1 - f_{\mathbf{p},\alpha}) \right] \ .$$

$$\mathbf{j} = -ev_F \sum_{\mathbf{p},\alpha} \left[\hat{u}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p},\alpha} + \hat{v}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{v}_\alpha \right] (1 - 2f_{\mathbf{p},\alpha})$$



(a) (b) (c)

Unit cell | Brillouin zone | Extended zone scheme

$$\mathbf{j} = \int \frac{d^2p}{(2\pi)^2} \left[\mathbf{j}_{\mathbf{K}}(\mathbf{p}) + \mathbf{j}_{-\mathbf{K}}(\mathbf{p}) \right]$$

$$\mathbf{j}_{\mathbf{K}}(\mathbf{p}) = -ev_F \sum_{\alpha=1}^{4} \hat{u}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{u}_{\mathbf{p},\alpha} \left[1 - 2f_{\mathbf{p},\alpha} \right]$$

$$\mathbf{j}_{-\mathbf{K}}(\mathbf{p}) = -ev_F \sum_{\alpha=1}^{4} \hat{v}^\dagger_{\mathbf{p},\alpha} \hat{\boldsymbol{\sigma}} \hat{v}_{\mathbf{p},\alpha} \left[1 - 2f_{\mathbf{p},\alpha} \right]$$

Zero-current ground state

Define spinors that satisfy

$$(\hat{\boldsymbol{\sigma}} \cdot \mathbf{p})\hat{a}_{\uparrow,\downarrow} = \pm p\, \hat{a}_{\uparrow,\downarrow}$$

The states with pseudospin parallel and anti-parallel to the momentum

$$\hat{a}_{\uparrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - i p_y}{p}} \\ \sqrt{\frac{p_x + i p_y}{p}} \end{pmatrix} \;,\; \hat{a}_{\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{p_x - i p_y}{p}} \\ -\sqrt{\frac{p_x + i p_y}{p}} \end{pmatrix}$$

The spinors $\hat{a}_{\uparrow}$ and $\hat{a}_{\downarrow}$ are eigenstates of excitations in the normal graphene.

We introduce vectors in the Nambu space,

$$\check{\psi} = \begin{pmatrix} \hat{u} \\ \hat{v} \end{pmatrix} \;,\; \check{\psi}^{+} = (\hat{u}^{\dagger} \;,\; \hat{v}^{\dagger}) \;,\; \check{\psi}_{\alpha}^{+}\check{\psi}_{\beta} = \delta_{\alpha\beta}$$

Eigen-states for zero current

For $\uparrow$ spin

$$E_{1,2}^{(0)} = \pm E_\uparrow \ , \ E_\uparrow = \sqrt{(v_F p - \mu)^2 + |\Delta|^2}$$

$$\begin{pmatrix} \hat{u}_1^{(0)} \\ \hat{v}_1^{(0)} \end{pmatrix} = \begin{pmatrix} u_\uparrow \\ v_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p}\cdot\mathbf{r}}, \begin{pmatrix} \hat{u}_2^{(0)} \\ \hat{v}_2^{(0)} \end{pmatrix} = \begin{pmatrix} v_\uparrow \\ -u_\uparrow \end{pmatrix} \hat{a}_\uparrow e^{i\mathbf{p}\cdot\mathbf{r}}$$

For $\downarrow$ spin

$$E_{3,4}^{(0)} = \pm E_\downarrow \ , \ E_\downarrow = \sqrt{(v_F p + \mu)^2 + |\Delta|^2}$$

$$\begin{pmatrix} \hat{u}_3^{(0)} \\ \hat{v}_3^{(0)} \end{pmatrix} = \begin{pmatrix} u_\downarrow \\ v_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p}\cdot\mathbf{r}}, \begin{pmatrix} \hat{u}_4^{(0)} \\ \hat{v}_4^{(0)} \end{pmatrix} = \begin{pmatrix} v_\downarrow \\ -u_\downarrow \end{pmatrix} \hat{a}_\downarrow e^{i\mathbf{p}\cdot\mathbf{r}}.$$

$$u_\uparrow = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{v_F p - \mu}{E_\uparrow}} \quad , \quad v_\uparrow = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{v_F p - \mu}{E_\uparrow}} \ ,$$

$$u_\downarrow = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{v_F p + \mu}{E_\downarrow}} \quad , \quad v_\downarrow = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{v_F p + \mu}{E_\downarrow}}$$

Current-carrying state

Spectrum

$$
\begin{aligned}
(E^2-\mu^2)^2 - 2|\Delta|^2(E^2-\mu^2) + |\Delta|^4 + 2|\Delta|^2 v_F^2 \mathbf{p}_+\mathbf{p}_- & \\
-(E+\mu)^2 v_F^2 \mathbf{p}_-^2 - (E-\mu)^2 v_F^2 \mathbf{p}_+^2 + v_F^4 \mathbf{p}_+^2 \mathbf{p}_-^2 &= 0
\end{aligned}
$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$.

Two limiting cases

- $v_F k \ll \mu$

$$E_\alpha = E_\alpha^{(0)} + E_\alpha^{(1)}$$

$$E_{1,2}^{(1)} = -E_{3,4}^{(1)} = E_D \ , \ E_D = v_F(\mathbf{p}\cdot\mathbf{k})/2p$$

Doppler-shifted energies

- $\mu = 0$:

$$E_\pm^2 = |\Delta|^2 + v_F^2(p^2 + k^2/4) \pm \sqrt{|\Delta|^2 v_F^2 k^2 + v_F^4(\mathbf{p}\cdot\mathbf{k})^2}$$

No Doppler shift

Degenerate state, $\boldsymbol{E_+^2 = E_-^2}$ **for** $\mathbf{k = 0}$

Linear response, $v_F k \ll \mu$

$$\check{\psi}_\alpha = \check{\psi}_\alpha^{(0)} + \sum_{\beta\neq\alpha} B_{\alpha\beta}\check{\psi}_\beta^{(0)} \ , \ B_{\alpha\beta} = \frac{v_F\, \check{\psi}_\beta^{(0)+}(\hat{\boldsymbol{\sigma}}\cdot\mathbf{k})\check{\psi}_\alpha^{(0)}}{2(E_\alpha^{(0)} - E_\beta^{(0)})} \ , \ B_{\beta\alpha} = -B_{\alpha\beta}^*$$

$$B_{12} = B_{21} = B_{34} = B_{43} = 0$$

$$\begin{aligned} B_{13} &= -B_{24} = -\frac{iv_F([\mathbf{p}\times\mathbf{k}]\cdot\mathbf{z})}{2p}\frac{(u_\downarrow^* u_\uparrow + v_\downarrow^* v_\uparrow)}{E_\uparrow - E_\downarrow} \ , \\ B_{23} &= B_{14} = \frac{iv_F([\mathbf{p}\times\mathbf{k}]\cdot\mathbf{z})}{2p}\frac{(u_\downarrow^* v_\uparrow - v_\downarrow^* u_\uparrow)}{E_\uparrow + E_\downarrow} \ . \end{aligned}$$

Supercurrent

$$\begin{aligned} \mathbf{j} &= -ev_F \sum_{\alpha,\mathbf{p}} \left[\hat{u}_\alpha^{(0)\dagger}\hat{\boldsymbol{\sigma}}\hat{u}_\alpha^{(0)} + \hat{v}_\alpha^{(0)\dagger}\hat{\boldsymbol{\sigma}}\hat{v}_\alpha^{(0)}\right]\left[1 - 2f(E_\alpha^{(0)} + E_\alpha^{(1)})\right] \\ &\quad -2ev_F \mathrm{Re} \sum_{\alpha\neq\beta,\mathbf{p}} B_{\alpha\beta}\left[\hat{u}_\alpha^{(0)\dagger}\hat{\boldsymbol{\sigma}}\hat{u}_\beta^{(0)} + \hat{v}_\alpha^{(0)\dagger}\hat{\boldsymbol{\sigma}}\hat{v}_\beta^{(0)}\right]\left[1 - 2f(E_\alpha^{(0)})\right] \ . \end{aligned}$$

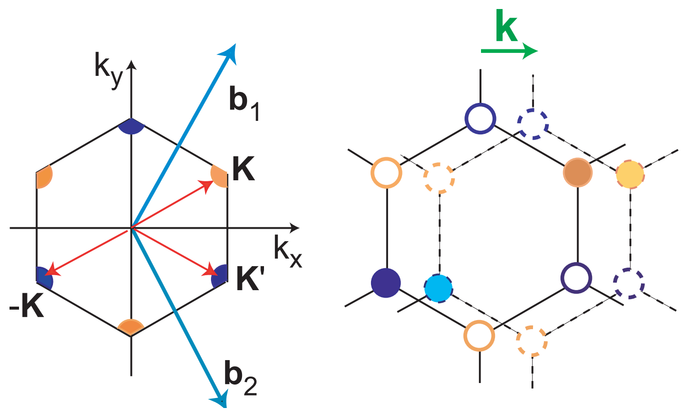
Correction to the supercurrent diverges because it extends over the entire BZ

Regularization of the divergence

The current-carrying state

$$\hat{u}_{\mathbf{p}} = \hat{u} e^{i(\mathbf{p}+\mathbf{k}/2)\cdot\mathbf{r}} \ , \ \hat{v}_{\mathbf{p}} = \hat{v} e^{i(\mathbf{p}-\mathbf{k}/2)\cdot\mathbf{r}} \ ,$$

contains contributions from the overall momentum shift in the BZ.



Since $v_{\mathbf{K}}(\mathbf{p}) = u^*_{-\mathbf{K}}(-\mathbf{p})$, a homogeneous shift by $\mathbf{k}$ gives $\mathbf{p}' \to \mathbf{p}' + \mathbf{k}$ and

$$u_{\mathbf{K}}(\mathbf{p}) \to u_{\mathbf{K}}(\mathbf{p}+\mathbf{k}) \ , \ u_{-\mathbf{K}}(-\mathbf{p}) \to u_{-\mathbf{K}}(-\mathbf{p}+\mathbf{k})$$

Therefore,

$$u_{\mathbf{K}}(\mathbf{p}) \to u_{\mathbf{K}}(\mathbf{p}+\mathbf{k}) \ , \ v_{\mathbf{K}}(\mathbf{p}) \to v_{\mathbf{K}}(\mathbf{p}-\mathbf{k})$$

Making shift of integration variable over the BZ $\mathbf{p} = \mathbf{p}' - \mathbf{k}/2$ in the zero-order current

$$\mathbf{j}^{(0)} = \int \frac{d^2p'}{(2\pi)^2} \left[\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p}' - \mathbf{k}/2) + \mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p}' + \mathbf{k}/2)\right]$$
$$= \int \frac{d^2p}{(2\pi)^2} \left[\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p}) + \mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p}) - \left(\mathbf{k}\cdot\frac{\partial}{\partial \mathbf{p}}\right)\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})\right] .$$

Here $\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p}) + \mathbf{j}^{(0)}_{-\mathbf{K}}(\mathbf{p}) = 0$. As a result

$$\mathbf{j}^{(0)} = -\int \frac{d\phi}{(2\pi)^2} [(\mathbf{p}\cdot\mathbf{k})\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})]_{p\gg\Delta,T} .$$

$$\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p}) = -ev_F\left[\frac{v_Fp-\mu}{E_\uparrow}\tanh\frac{E_\uparrow}{2T} + \frac{v_Fp+\mu}{E_\downarrow}\tanh\frac{E_\downarrow}{2T}\right]\frac{\mathbf{p}}{p}$$

$$\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})\Big|_{p\to\infty} = -2ev_F\left[1+\frac{\Delta^2}{2v_F^2p^2}\right]\frac{\mathbf{p}}{p} \to -2ev_F\frac{\mathbf{p}}{p}$$

and

$$(\mathbf{p}\cdot\mathbf{k})\,\mathbf{j}^{(0)}_{\mathbf{K}}(\mathbf{p})\Big|_{p\to\infty} \to -2ev_F\frac{\mathbf{p}}{p}(\mathbf{p}\cdot\mathbf{k})$$

The same result is obtained if one subtracts the normal current

For $T = 0$ the supercurrent becomes

$$\mathbf{j} = \frac{e\mathbf{k}}{2\pi}\left[\sqrt{\mu^2 + |\Delta|^2} + \frac{|\Delta|^2}{|\mu|}\ln\left(\frac{|\mu| + \sqrt{\mu^2 + |\Delta|^2}}{|\Delta|}\right)\right]$$

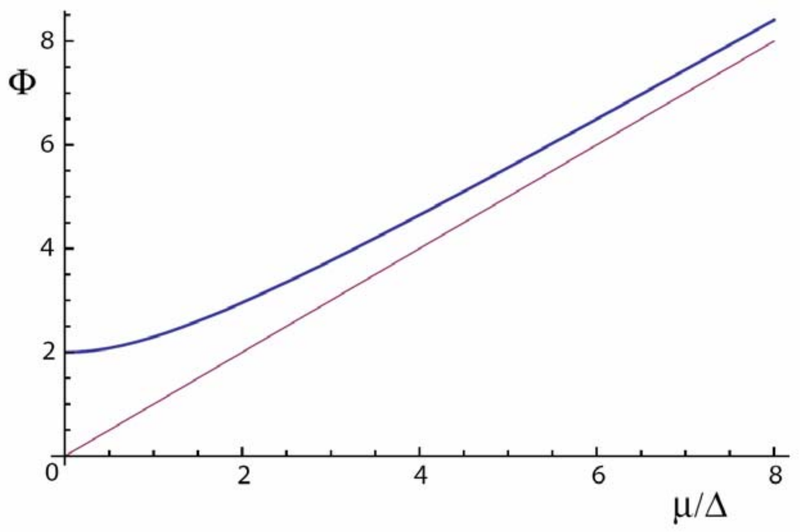
- For $\mu \gg |\Delta|$

$$\mathbf{j} = e|\mu|\mathbf{k}/2\pi$$

- For $\mu \ll |\Delta|$

$$\mathbf{j} = e|\Delta|\mathbf{k}/\pi$$

$$j = \frac{e|\Delta|\mathbf{k}}{2\pi}\Phi\left(\frac{\mu}{|\Delta|}\right)$$



This result formally holds within the linear approximation which assumes $v_F k \ll \mu$. Therefore, one has to put $k \to 0$ first and then assume $\mu \ll |\Delta|$.

What if $\mu \ll v_F k$?

The spectrum for $\mu = 0$:

$$E_\pm^2 = |\Delta|^2 + v_F^2(p^2 + k^2/4) \pm \sqrt{|\Delta|^2 v_F^2 k^2 + v_F^4(\mathbf{p}\cdot\mathbf{k})^2}$$

No Doppler energy

The zero-current state is degenerate:

$$E_\uparrow = E_\downarrow = E_0 = \sqrt{v_F^2 p^2 + |\Delta|^2}$$

$$E_1 = E_3 = E_0 \ , \ E_2 = E_4 = -E_0$$

Requires a special consideration

One can show that the linear-response results $v_F k \ll |\Delta|$ holds irrespectively of the relation between $v_F k$ and μ.

Conclusions

- No qualitative difference between the critical temperature, superconducting gap and supercurrent obtained for the simple model and for the two-valley BdG-Dirac description

  - **Slightly different parametric dependence of supercurrent**
  - **The $\mu \gg |\Delta|$ result gives 2 times larger current than in the simple model due to two times larger number of cones.**
  - **However, the $\mu \ll |\Delta|$ limit gives 4 times larger current. This is due to more subtle differences originating from interference of four ground states.**

- The supercurrent is finite at any doping level as long as superconductivity exists