On the Applicability of the Kinetic Equation for Waves to the Phillips Spectrum.

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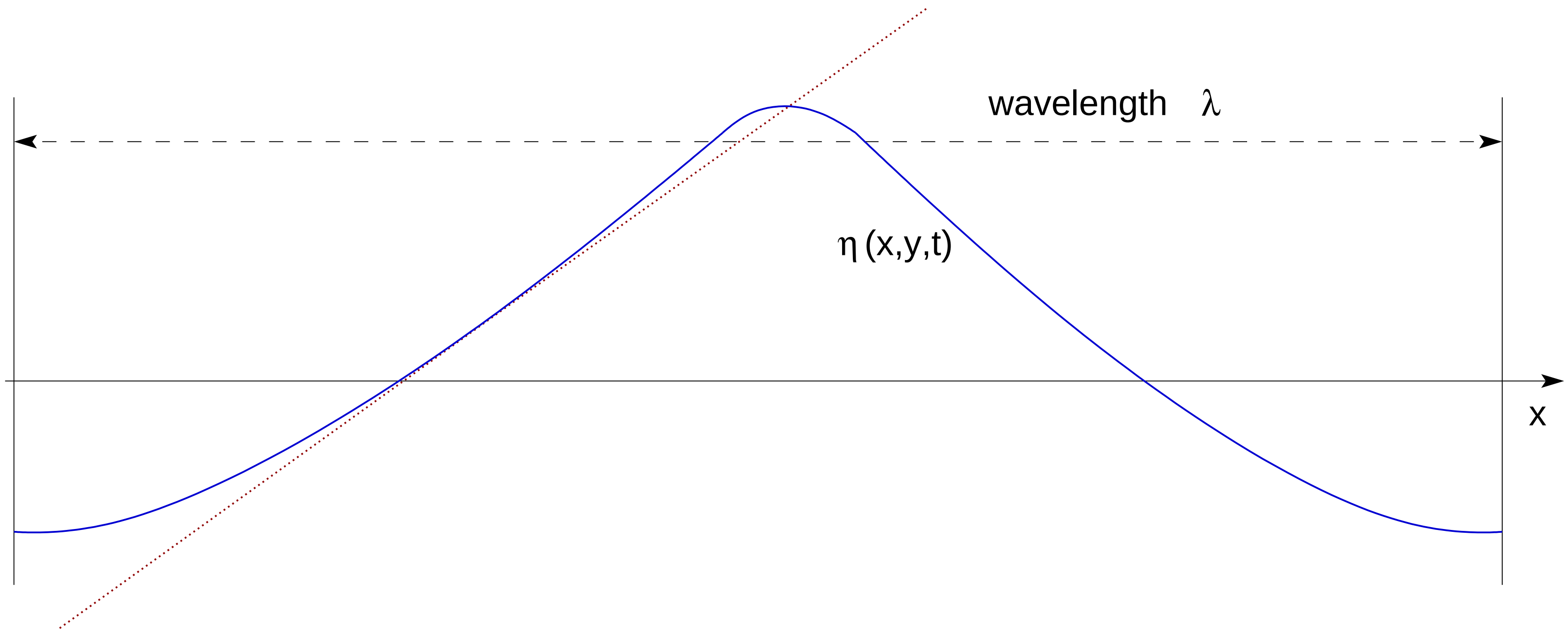
Problem formulation

Let us consider a potential flow of an ideal fluid of infinite depth with a free surface. We use standard notations for velocity potential $\phi(\vec{r}, z, t), \vec{r} = (x, y); \vec{v} = \nabla\phi$ and surface elevation $\eta(\vec{r}, t)$.



Steepness of the surface $\mu = \langle|\nabla\eta(\vec{r}, t)|\rangle$ — average slope of the surface.

Energy of the system

Fluid flow is incompressible $(\nabla\vec{v}) = \Delta\phi = 0$. The total energy of the system can be presented in the following form

$$H = T + U,$$

Kinetic energy:

$$T = \frac{1}{2}\int \mathrm{d}^2 r \int\limits_{-\infty}^{\eta} (\nabla\phi)^2 \mathrm{d}z, \tag{1}$$

Potential energy due to gravity:

$$U = \frac{1}{2} g \int \eta^2 \mathrm{d}^2 r, \tag{2}$$

here g is the gravity acceleration.

Hamiltonian expansion

It was shown by Zakharov (1966) that under these assumptions the fluid is a Hamiltonian system

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}, \tag{3}$$

where $\psi = \phi(\vec{r}, \eta(\vec{r}, t), t)$ is a velocity potential on the surface of the fluid. In order to calculate the value of ψ we have to solve the Laplace equation in the domain with varying surface η. One can simplify the situation, using the expansion of the Hamiltonian in powers of "steepness" (here $\Delta = \nabla^2$ and $\hat{k} = \sqrt{-\Delta}$)

$$\begin{aligned} H = &\frac{1}{2}\int \left(g\eta^2 + \psi\hat{k}\psi\right) \mathrm{d}^2 r + \\ +&\frac{1}{2}\int \eta\left[|\nabla\psi|^2 - (\hat{k}\psi)^2\right] \mathrm{d}^2 r + \\ +&\frac{1}{2}\int \eta(\hat{k}\psi)\left[\hat{k}(\eta(\hat{k}\psi)) + \eta\Delta\psi\right] \mathrm{d}^2 r. \end{aligned} \tag{4}$$

Dynamical equations

In this case dynamical equations acquire the following form

$$
\begin{aligned}
\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\
&+ \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi] - D_{\vec{r}}, \\
\dot{\psi} &= -g\eta - \tfrac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\
&- [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - D_{\vec{r}} + F_{\vec{r}}.
\end{aligned}
\tag{5}
$$

Here $D_{\vec{r}}$ is some artificial damping term used to provide dissipation at small scales; $F_{\vec{r}}$ is a pumping term corresponding to external force (having in mind wind blow, for example). Let us introduce Fourier transform

$$
\psi_{\vec{k}} = \frac{1}{2\pi}\int \psi_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r, \quad \eta_{\vec{k}} = \frac{1}{2\pi}\int \eta_{\vec{r}} \mathrm{e}^{\mathrm{i}\vec{k}\vec{r}} \mathrm{d}^2 r.
$$

Canonical variables

$\psi(\vec{r},t)$ and $\eta(\vec{r},t)$ are real valued functions, $\Rightarrow \psi_{\vec{k}} = \psi^*_{-\vec{k}}, \eta_{\vec{k}} = \eta^*_{-\vec{k}}$ — Hermitian symmetry.

It is convenient to introduce the canonical (normal) variables $a_{\vec{k}}$ as shown below

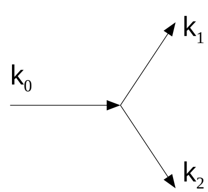
$$a_{\vec{k}} = \sqrt{\frac{\omega_k}{2k}}\eta_{\vec{k}} + \mathrm{i}\sqrt{\frac{k}{2\omega_k}}\psi_{\vec{k}}, \text{ where } \omega_k = \sqrt{gk}.$$

$$\dot{a}_{\vec{k}} = -\mathrm{i}\frac{\delta H}{\delta a^*_{\vec{k}}} \text{ — Hamiltonian equations,}$$

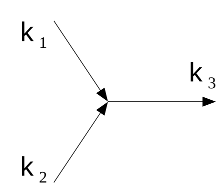
$a_{\vec{k}}$ — is an elementary excitation (plane wave).

$$
\begin{aligned}
H_0 &= \int \omega_k |a_{\vec{k}}|^2 \mathrm{d}\vec{k}, \\
H_1 &= \frac{1}{6}\frac{1}{2\pi}\int E^{\vec{k}_0}_{\vec{k}_1\vec{k}_2}(a_{\vec{k}_1}a_{\vec{k}_2}a_{\vec{k}_0} + a^*_{\vec{k}_1}a^*_{\vec{k}_2}a^*_{\vec{k}_0})\delta(\vec{k}_1+\vec{k}_2+\vec{k}_0)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_0 + \\
&+\frac{1}{2}\frac{1}{2\pi}\int C^{\vec{k}_0}_{\vec{k}_1\vec{k}_2}(a_{\vec{k}_1}a_{\vec{k}_2}a^*_{\vec{k}_0} + a^*_{\vec{k}_1}a^*_{\vec{k}_2}a_{\vec{k}_0})\delta(\vec{k}_1+\vec{k}_2-\vec{k}_0)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_0, \\
H_2 &= \frac{1}{4}\frac{1}{(2\pi)^2}\int W_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}(a_{\vec{k}_1}a_{\vec{k}_2}a_{\vec{k}_3}a_{\vec{k}_4} + a^*_{\vec{k}_1}a^*_{\vec{k}_2}a^*_{\vec{k}_3}a^*_{\vec{k}_4})\times \\
&\times\delta(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\mathrm{d}\vec{k}_4 + \\
&+\frac{1}{4}\frac{1}{(2\pi)^2}\int F_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}(a^*_{\vec{k}_1}a_{\vec{k}_2}a_{\vec{k}_3}a_{\vec{k}_4} + a_{\vec{k}_1}a^*_{\vec{k}_2}a^*_{\vec{k}_3}a^*_{\vec{k}_4})\times \\
&\times\delta(\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\mathrm{d}\vec{k}_4 + \\
&+\frac{1}{4}\frac{1}{(2\pi)^2}\int D_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}(a_{\vec{k}_1}a_{\vec{k}_2}a^*_{\vec{k}_3}a^*_{\vec{k}_4}\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\mathrm{d}\vec{k}_4.
\end{aligned}
\tag{6}
$$

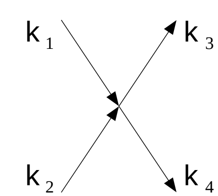
$a_{\vec{k}_0}a^*_{\vec{k}_1}a^*_{\vec{k}_2}$



$a_{\vec{k}_1}a_{\vec{k}_2}a^*_{\vec{k}_3}$



$a_{\vec{k}_1}a_{\vec{k}_2}a^*_{\vec{k}_3}a^*_{\vec{k}_4}$



Resonant conditions

Let us get rid of linear part:

$$(a_{\vec{k}_1} a_{\vec{k}_2} a^*_{\vec{k}_0} + a^*_{\vec{k}_1} a^*_{\vec{k}_2} a_{\vec{k}_0})\delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_0)$$

$$a_{\vec{k}}(t) = A_{\vec{k}}(t)\mathrm{e}^{\mathrm{i}\omega_k t} \Rightarrow a^*_{\vec{k}_0} a_{\vec{k}_1} a_{\vec{k}_2} = A^*_{\vec{k}_0} A_{\vec{k}_1} A_{\vec{k}_2} \mathrm{e}^{\mathrm{i}(\omega_{k_0} - \omega_{k_1} - \omega_{k_2})t}$$

Resonant conditions for 3-waves interaction (decaying and merging):

$$\omega_{k_0} = \omega_{k_1} + \omega_{k_2}, \quad \vec{k}_0 = \vec{k}_1 + \vec{k}_2.$$

Resonant conditions for 4-waves interaction (two into two scattering):

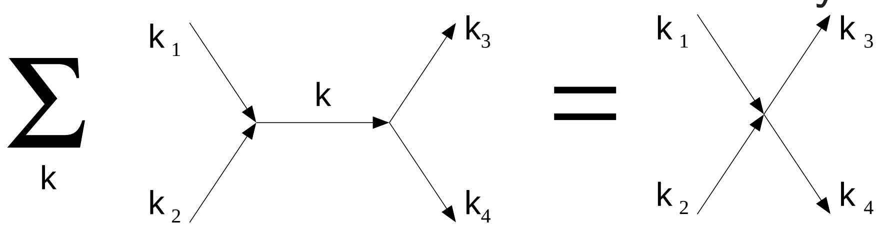
$$\omega_{k_1} + \omega_{k_2} = \omega_{k_3} + \omega_{k_4}, \quad \vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4. \tag{7}$$

Canonical transformation

The dispersion relation in the case of gravity waves on a deep water $\omega_k = \sqrt{gk}$ is of the "non-decay type" and equations

$$\omega_{k_1} = \omega_{k_2} + \omega_{k_3}, \quad \vec{k}_1 = \vec{k}_2 + \vec{k}_3 \tag{8}$$

have no real solution. It means that in the limit of small nonlinearity, the cubic terms in the Hamiltonian can be excluded by a canonical transformation $a(\vec{k}, t) \longrightarrow b(\vec{k}, t)$.



$$H_0 = \int \omega_k |b_{\vec{k}}|^2 \mathrm{d}\vec{k},$$
$$H_1 = 0,$$
$$H_2 = \frac{1}{2}\frac{1}{(2\pi)^2}\int T_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} b^*_{\vec{k}_1} b^*_{\vec{k}_2} b_{\vec{k}_3} b_{\vec{k}_4} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)\mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{k}_3\mathrm{d}\vec{k}_4.$$

Pair correlation functions

For statistical description of a stochastic wave field one can use a pair correlation function

$$\langle a_{\vec{k}} a^*_{\vec{k}'} \rangle = n_k \delta(\vec{k} - \vec{k}'). \tag{9}$$

The $n_{\vec{k}}$ is measurable quantity, connected directly with observable correlation functions. For instance, from $a_{\vec{k}}$ definition one can get

$$I_k = \langle |\eta_{\vec{k}}|^2 \rangle = \frac{1}{2} \frac{\omega_k}{g} (n_k + n_{-k}). \tag{10}$$

To derive kinetic equation in the case of gravity waves it is convenient to use another correlation function

$$\langle b_{\vec{k}} b^*_{\vec{k}'} \rangle = N_k \delta(\vec{k} - \vec{k}'). \tag{11}$$

Relation between correlation functions

The relation connecting n_k and N_k is very simple (in the case of deep water)

$$\frac{n_k - N_k}{n_k} \simeq \mu, \tag{12}$$

where $\mu = (ka)^2$, here a is a characteristic elevation of the free surface. In the case of the weak turbulence $\mu \ll 1$.

Kinetic equation

The correlation function N_k obeys the kinetic equation (Nordheim,1929; Hasselmann, 1962; Zakharov, 1966)

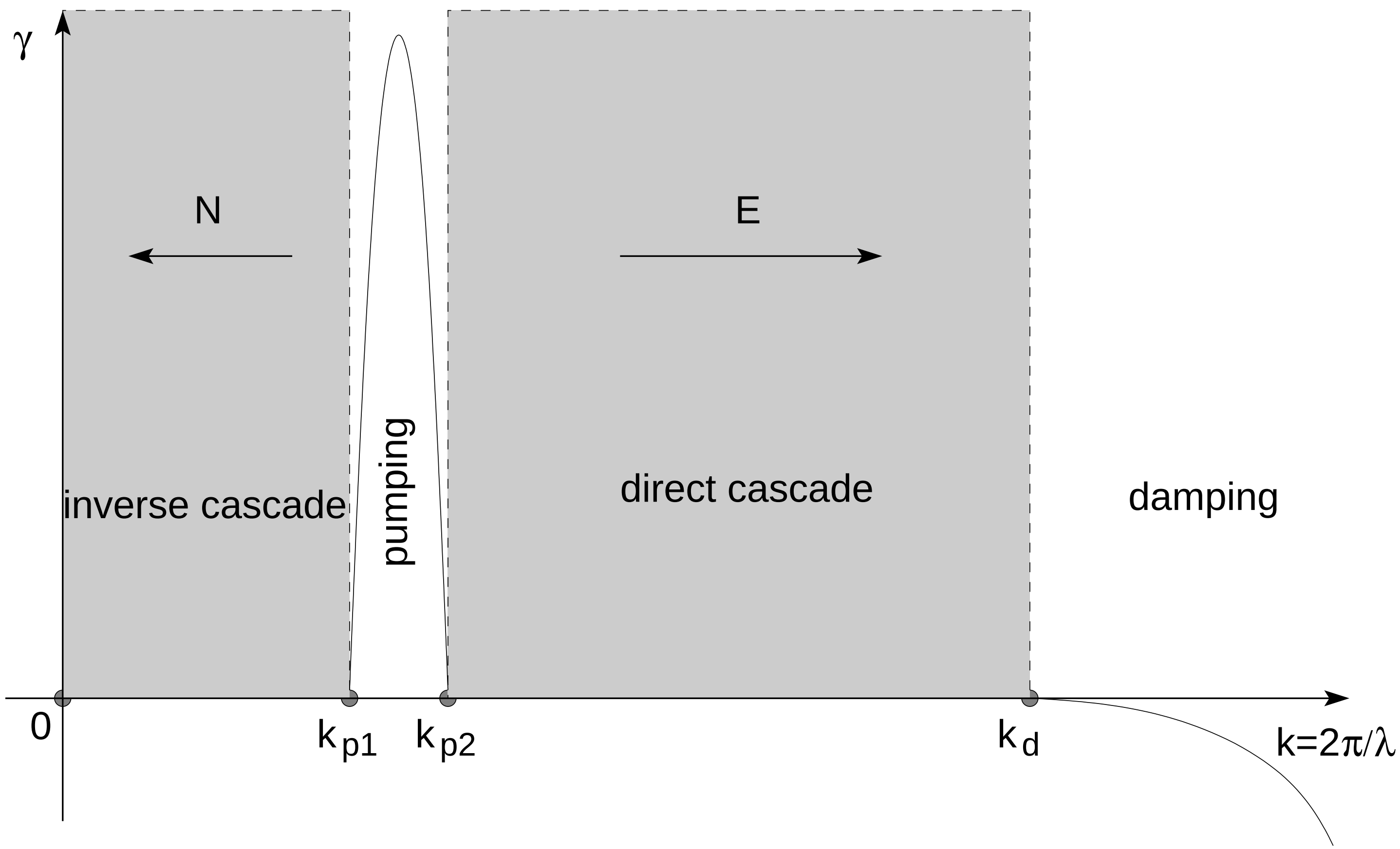
$$\frac{\partial N_k}{\partial t} = st(N, N, N) + f_p(k) - f_d(k), \qquad (13)$$

Here

$$\begin{aligned} st(N, N, N) = 4\pi \int \left| T_{\vec{k},\vec{k}_1,\vec{k}_2,\vec{k}_3} \right|^2 \times \\ \times (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - \\ -N_k N_{k_1} N_{k_3}) \delta(\vec{k} + \vec{k}_1 - \vec{k}_2 - \vec{k}_3) \delta(\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}) \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3. \end{aligned} \qquad (14)$$

The kinetic equation and its modifications are the base for all wave forecasting models.

Scheme of scales



Numerical scheme parameters

Let us add damping and pumping in dynamical equations

$$
\begin{aligned}
\dot{\eta} &= \hat{k}\psi - (\nabla(\eta\nabla\psi)) - \hat{k}[\eta\hat{k}\psi] + \\
&\quad + \hat{k}(\eta\hat{k}[\eta\hat{k}\psi]) + \tfrac{1}{2}\Delta[\eta^2\hat{k}\psi] + \tfrac{1}{2}\hat{k}[\eta^2\Delta\psi] - F^{-1}[\gamma_k\eta_{\vec{k}}], \\
\dot{\psi} &= -g\eta - \tfrac{1}{2}\left[(\nabla\psi)^2 - (\hat{k}\psi)^2\right] - \\
&\quad - [\hat{k}\psi]\hat{k}[\eta\hat{k}\psi] - [\eta\hat{k}\psi]\Delta\psi - F^{-1}[\gamma_k\psi_{\vec{k}}] + F^{-1}[f_k \mathrm{e}^{\mathrm{i}R_{\vec{k}}(t)}].
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&f_k = 4F_0\frac{(k-k_{p1})(k_{p2}-k)}{(k_{p2}-k_{p1})^2}; \\
&D_{\vec{k}} = \gamma_k\psi_{\vec{k}}, \\
&\gamma_k = \gamma_0(k-k_d)^2, k > k_d.
\end{aligned}
\tag{16}
$$

Here $R_{\vec{k}}(t)$ — uniformly distributed random number in interval $(0, 2\pi]$. Simulation region $L_x = L_y = 2\pi$ with double periodic boundary conditions. Pumping parameters: $F_0 = 1.5 \times 10^{-5}, k_{p1} = 28, k_{p2} = 32$.

Numerical algorithm: multigrid approach

$\omega_k = \sqrt{gk}$ — it means the larger is k (smaller scale), the smaller is characteristic stabilization time. So it is natural to stabilize large scales (small k), then increase grid and stabilize smaller scales (larger k).

Damping parameters for different grid sizes:

$$256 \times 256,\ k_d = 64,\ \gamma_0 = 3.09 \times 10^2,\ \Delta t = 5.00 \times 10^{-3}$$

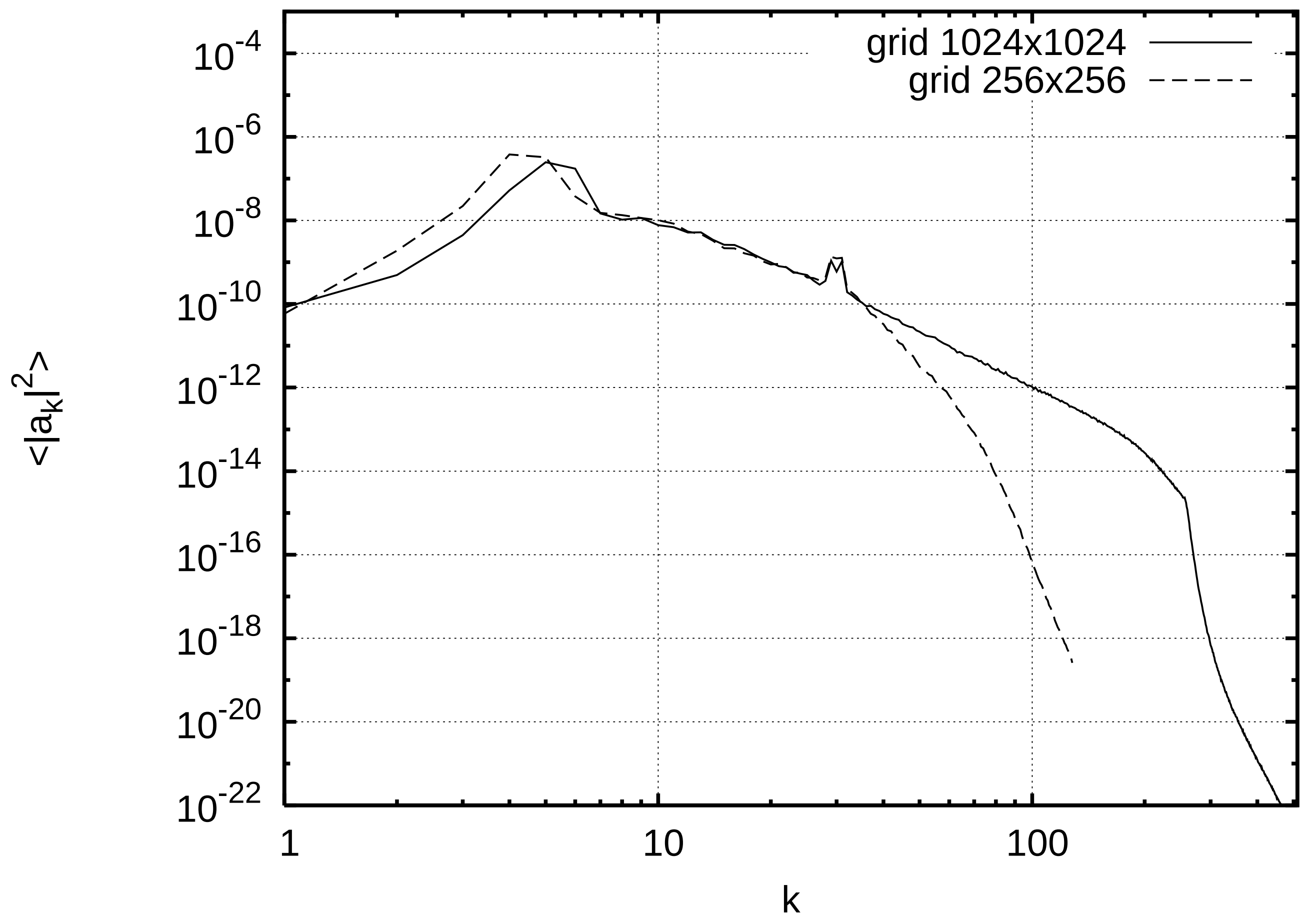
$$512 \times 512,\ k_d = 128,\ \gamma_0 = 5.54 \times 10^3,\ \Delta t = 2.12 \times 10^{-3}$$

$$1024 \times 1024,\ k_d = 256,\ \gamma_0 = 2.06 \times 10^4,\ \Delta t = 6.67 \times 10^{-4}$$

Numerical algorithm: operator splitting

- Solving conservative Hamiltonian equations by implicit scheme: discrete analog of Hamiltonian equations
- Control over conservation of Hamiltonian on one step up to desired accuracy
- Require fast convergence: control over nonlinearity related timestep restrictions
- Taking into account pumping and damping by exact integration of corresponding parts of equations

Spectra. Angle averaged.



Origin of the condensate

Resonant conditions are never fulfilled exactly on the discrete grid of wavevectors:

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} \neq 0.$$

Nonlinear frequency shift gives finite width of resonant curve:

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} \leq \Gamma.$$

Nonlinear frequency shift depends on $T_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4}$, which is homogeneous function. It grows as k^3 when k is increased. It also decreases as k^3 when $k \to 0$.

Inverse cascade is stopped because four-waves nonlinear interaction is “turned off” at some scale. At the same time flux still brings new waves to this scale. We have “condensation” of waves.

Zakharov-Kolmogorov solutions (deep water)

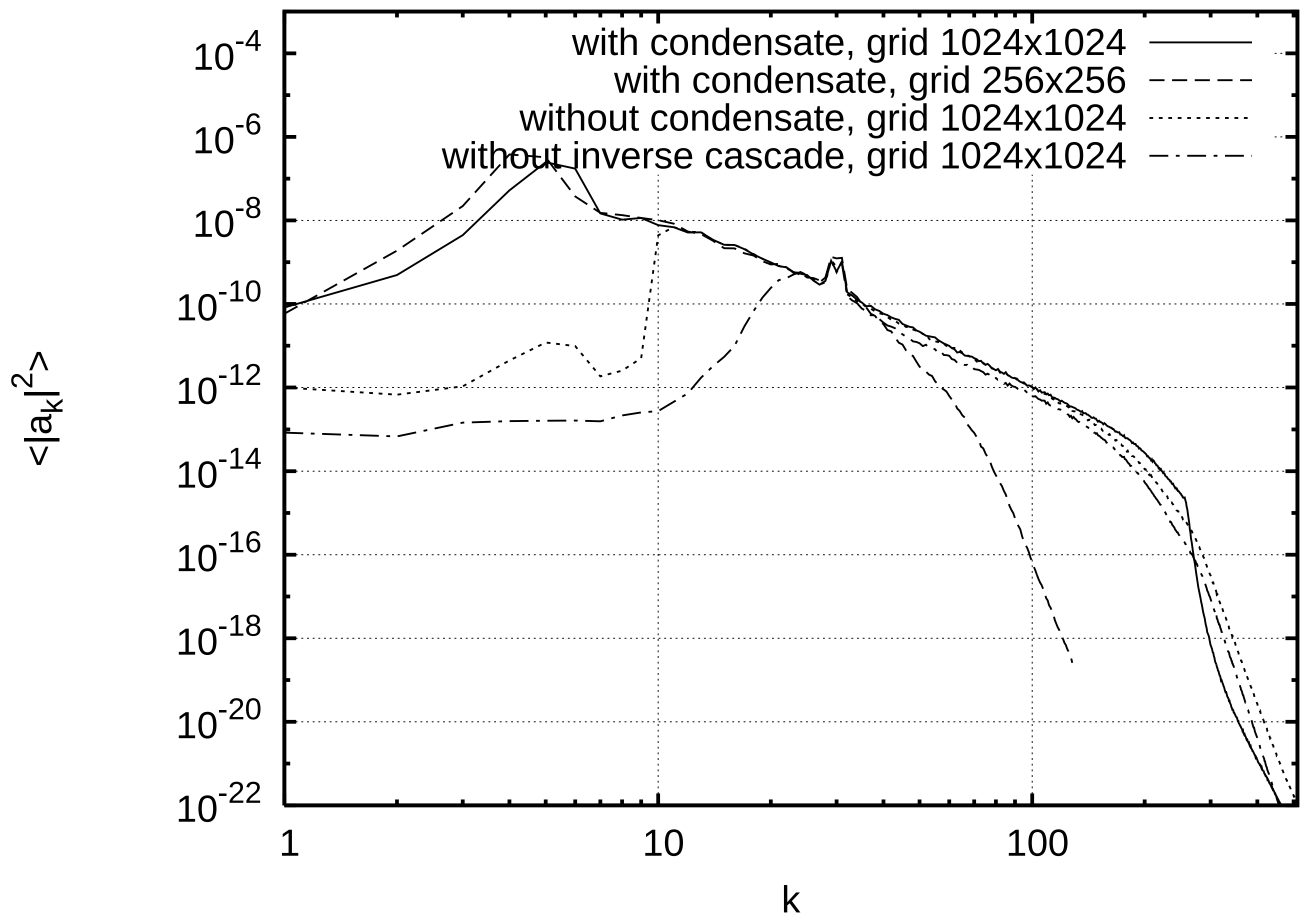
Direct cascade of energy from large scale to small scales (Zakharov and Filonenko, 1967)

$$n_k^{(1)} = C_1 P^{1/3} k^{-\frac{2\beta}{3}-d} = C_1 P^{1/3} k^{-4}. \tag{17}$$

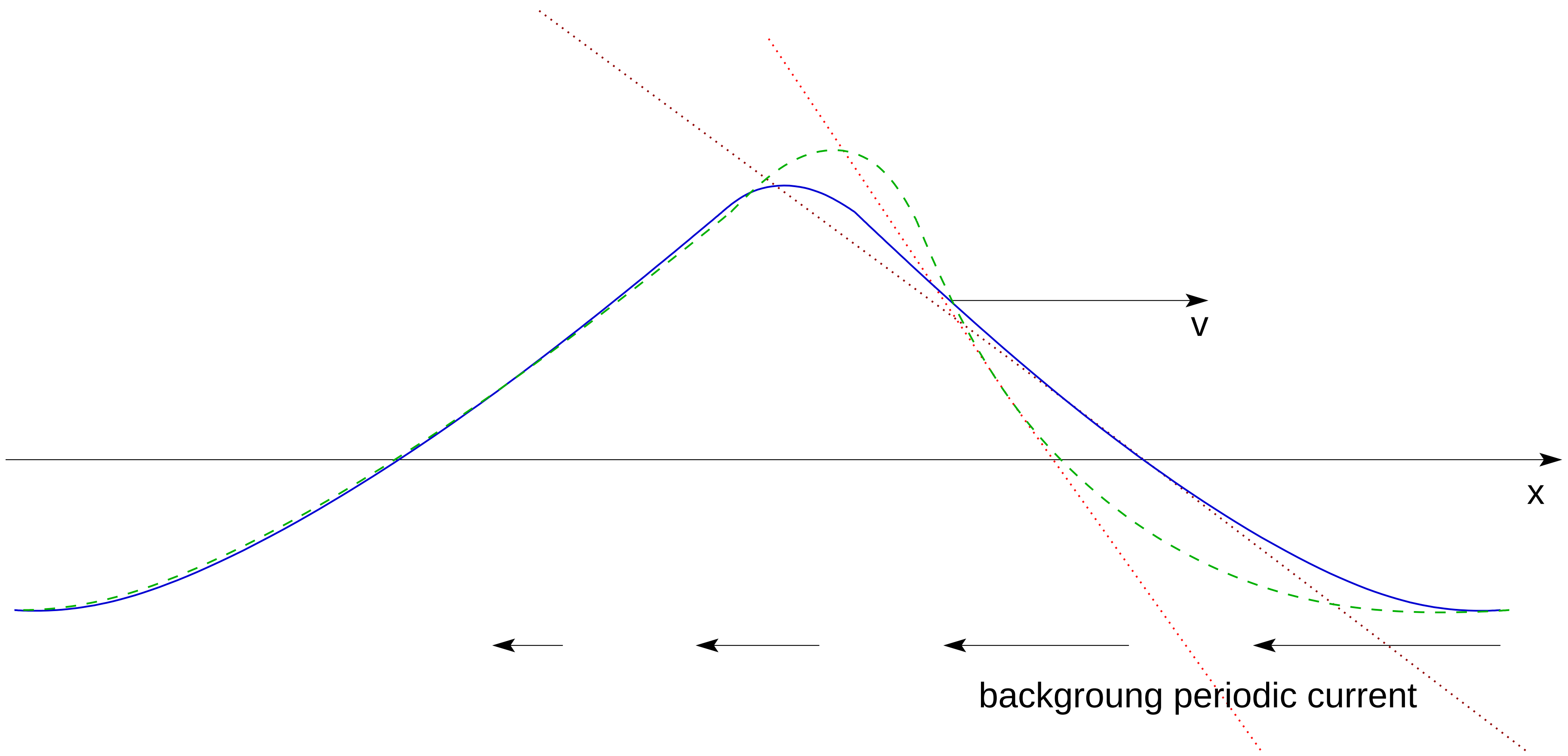
Inverse cascade of number of waves (Zakharov, 1968)

$$n_k^{(2)} = C_2 Q^{1/3} k^{-\frac{2\beta-\alpha}{3}-d} = C_2 Q^{1/3} k^{-23/6}. \tag{18}$$

Spectra. Angle averaged.

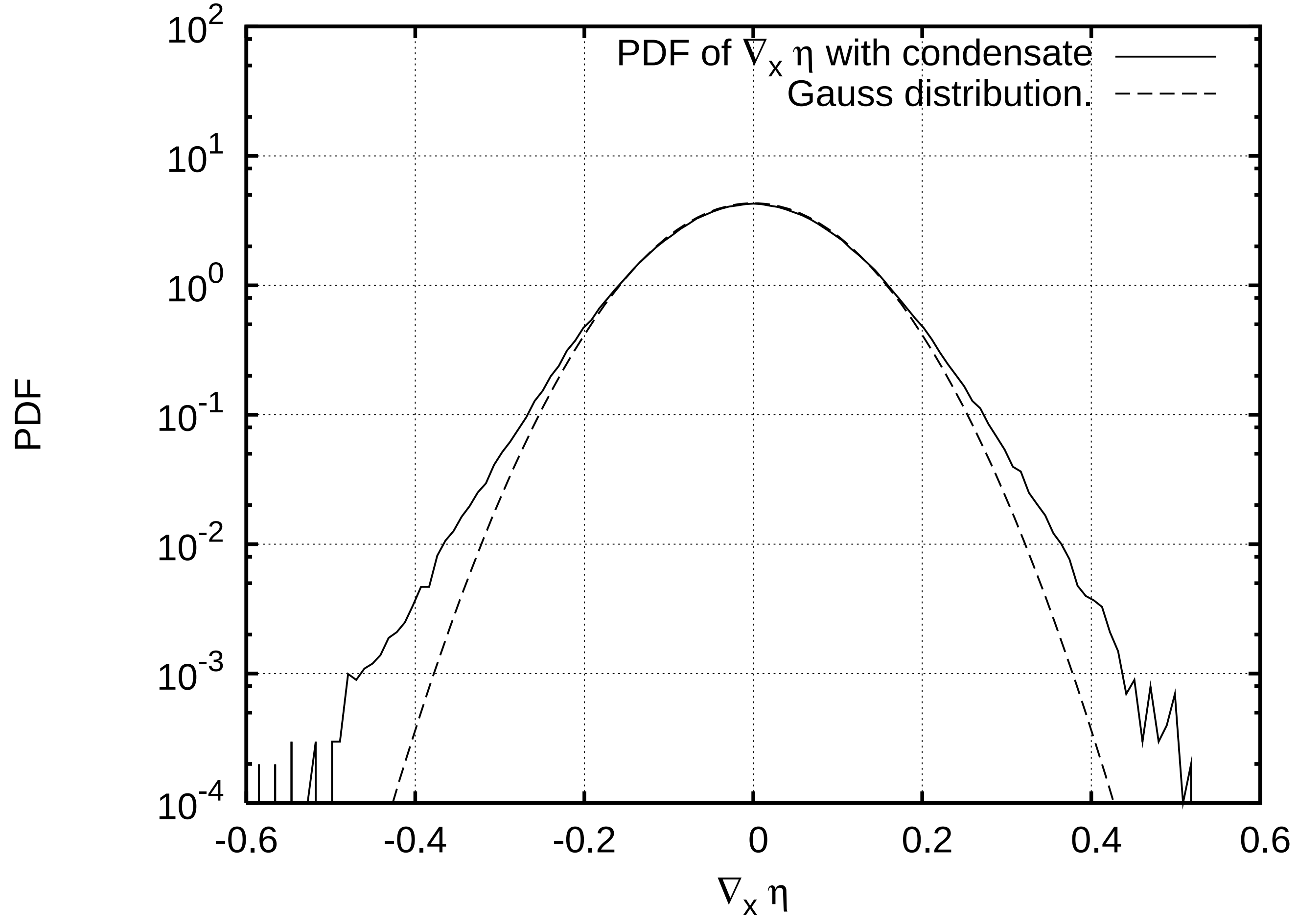


Qualitative picture: scheme

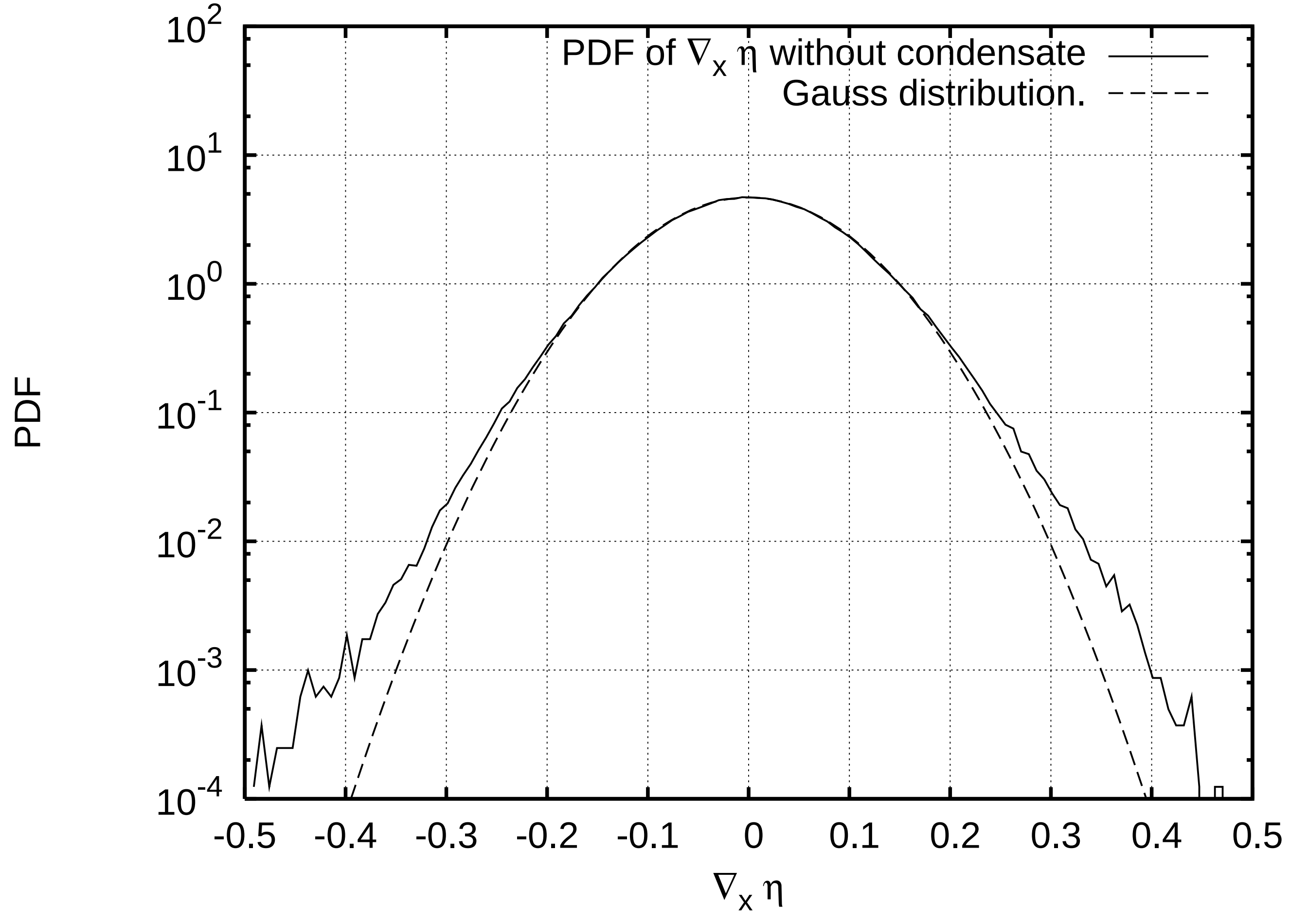


Qualitative picture: whitecapping

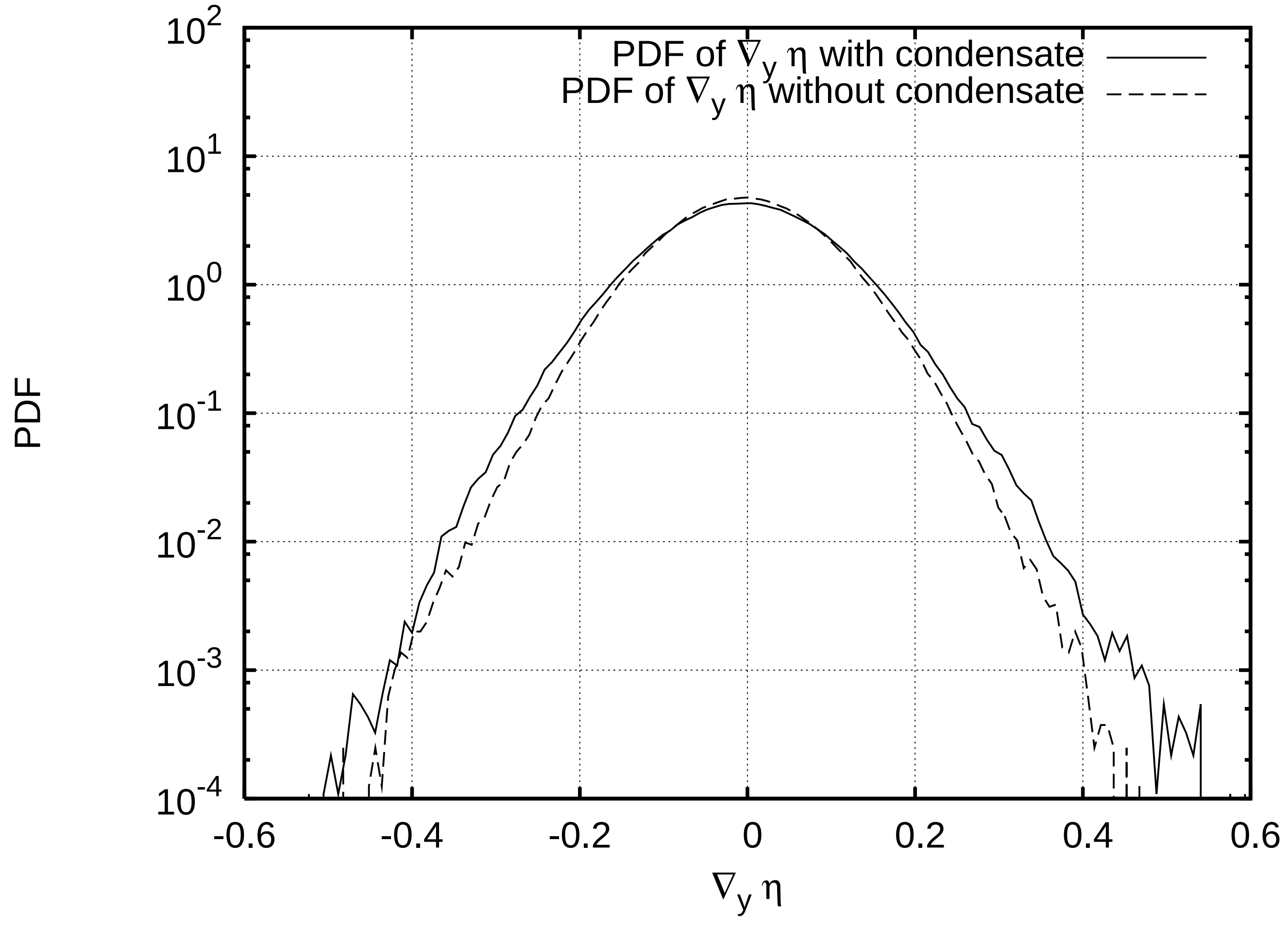
Surface. Gradient PDF. With condensate.



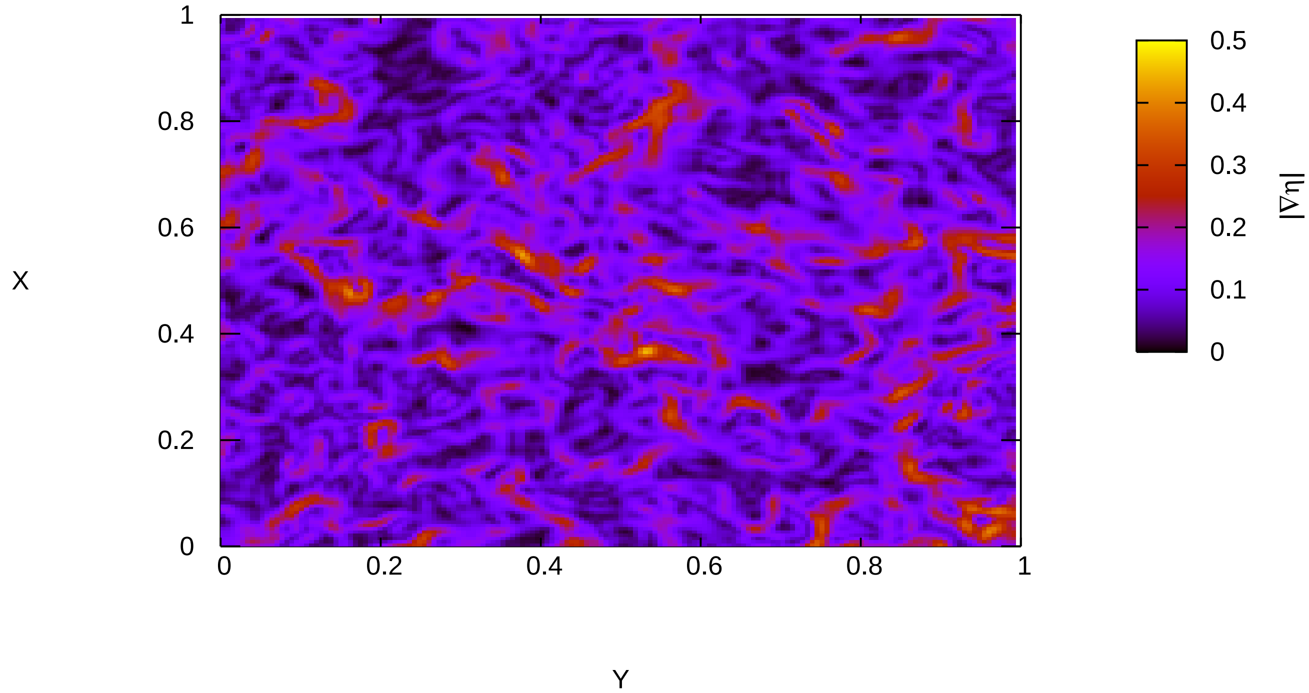
Surface. Gradient PDF. Without condensate.



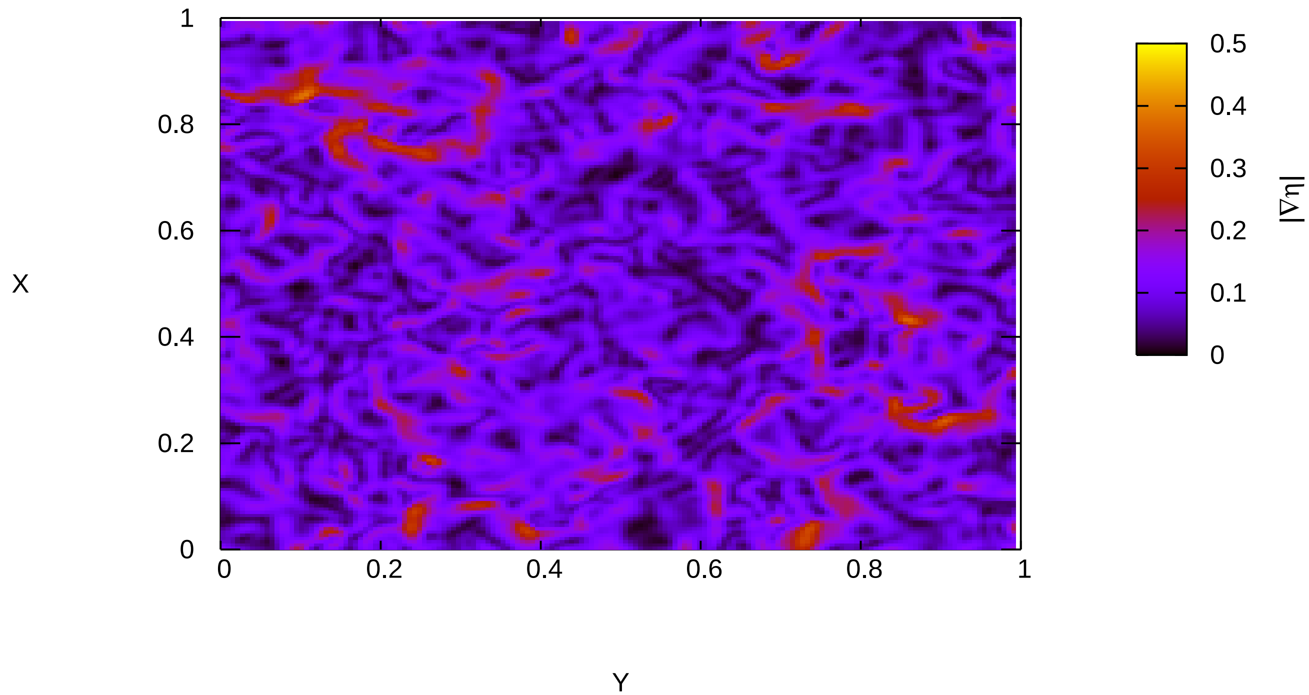
Surface. Gradient PDF. With and without condensate.



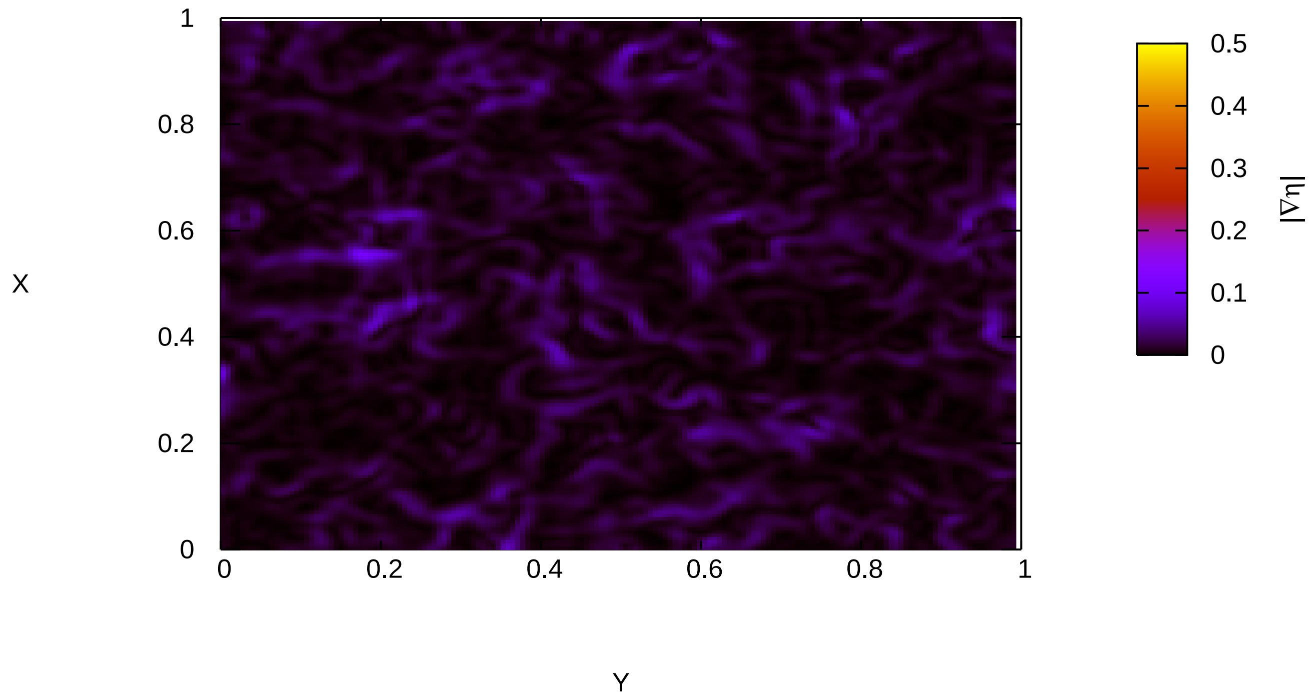
Surface. Gradient. With condensate.



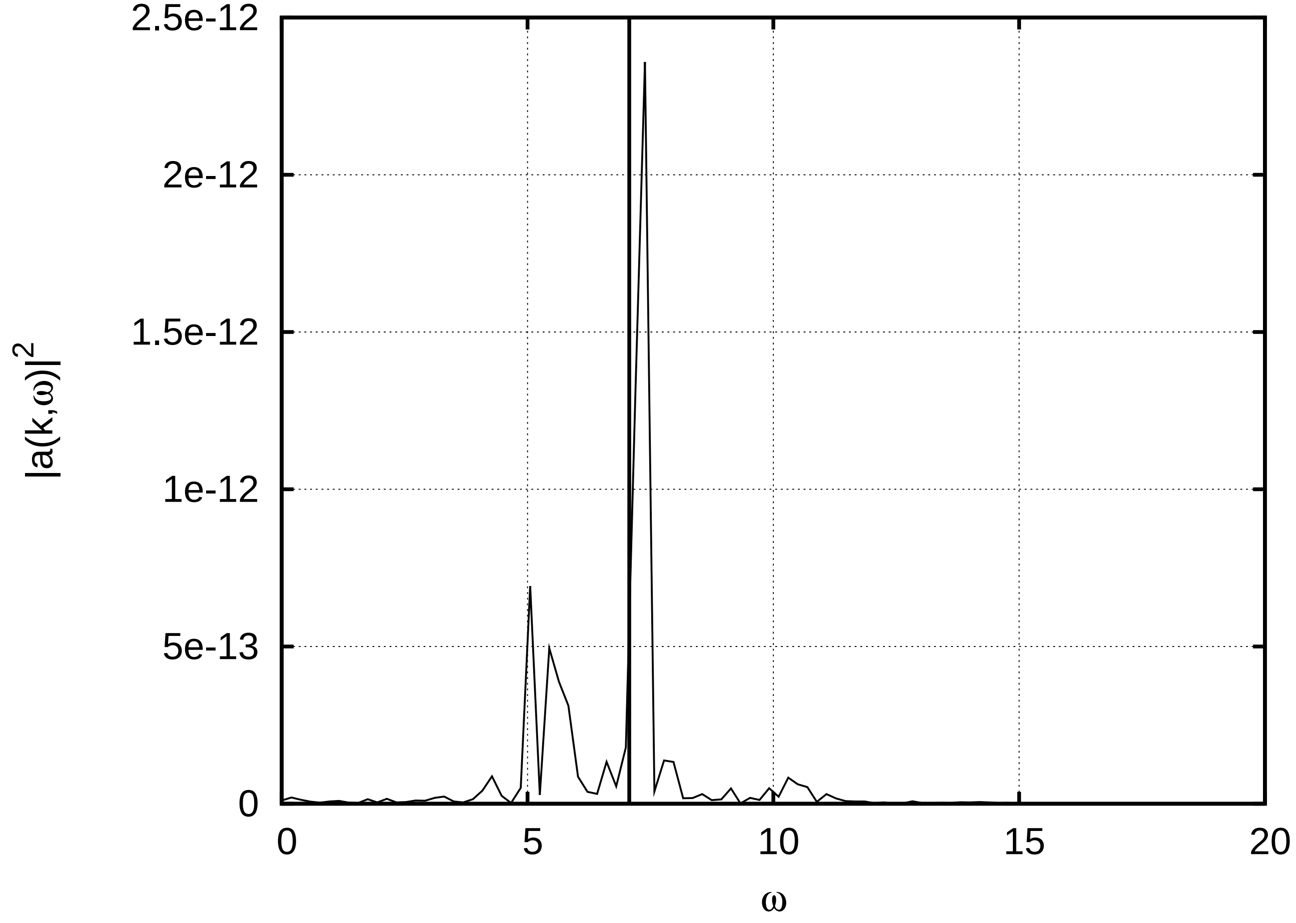
Surface. Gradient. Without condensate.



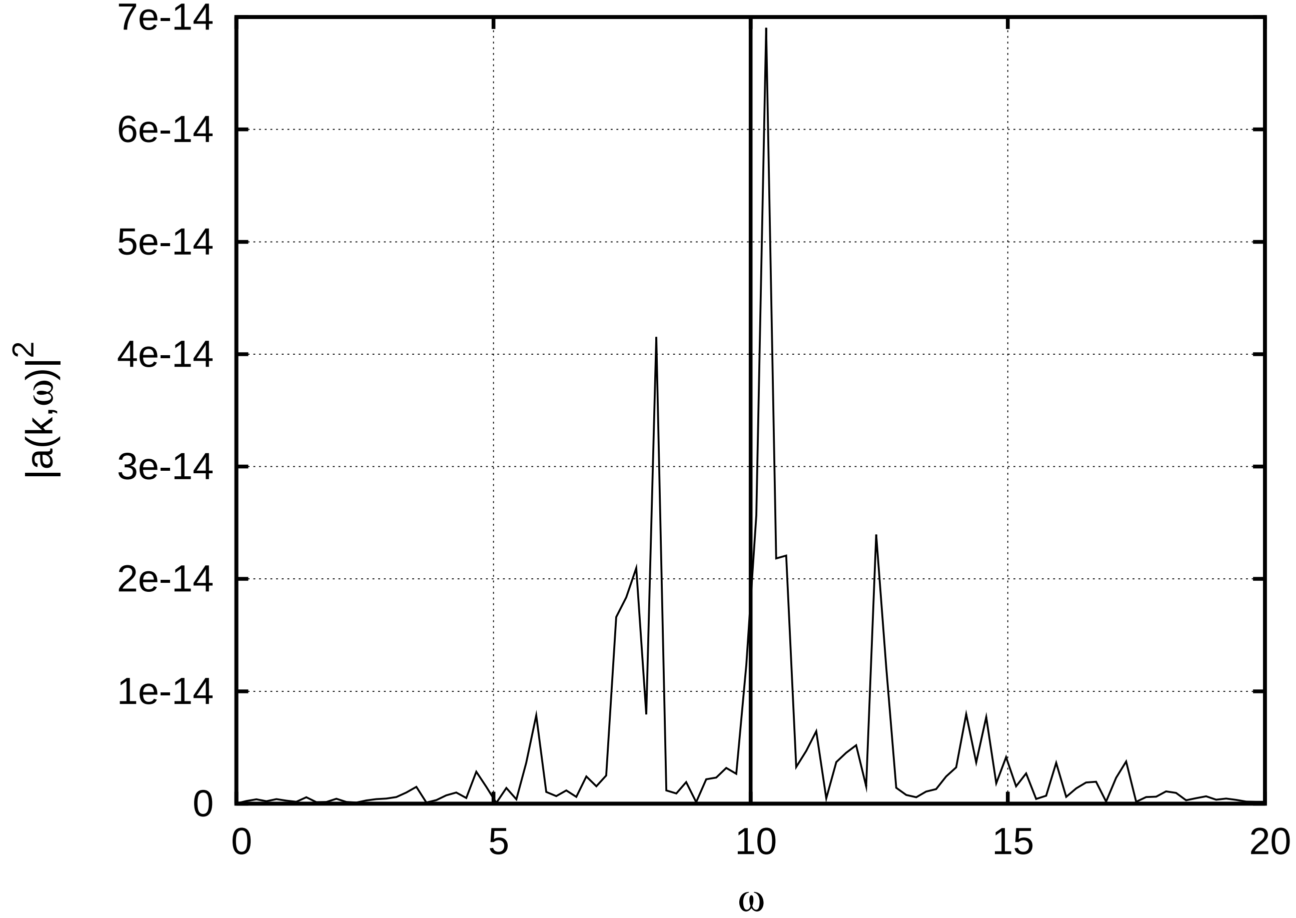
Surface. Gradient. Without inverse cascade.



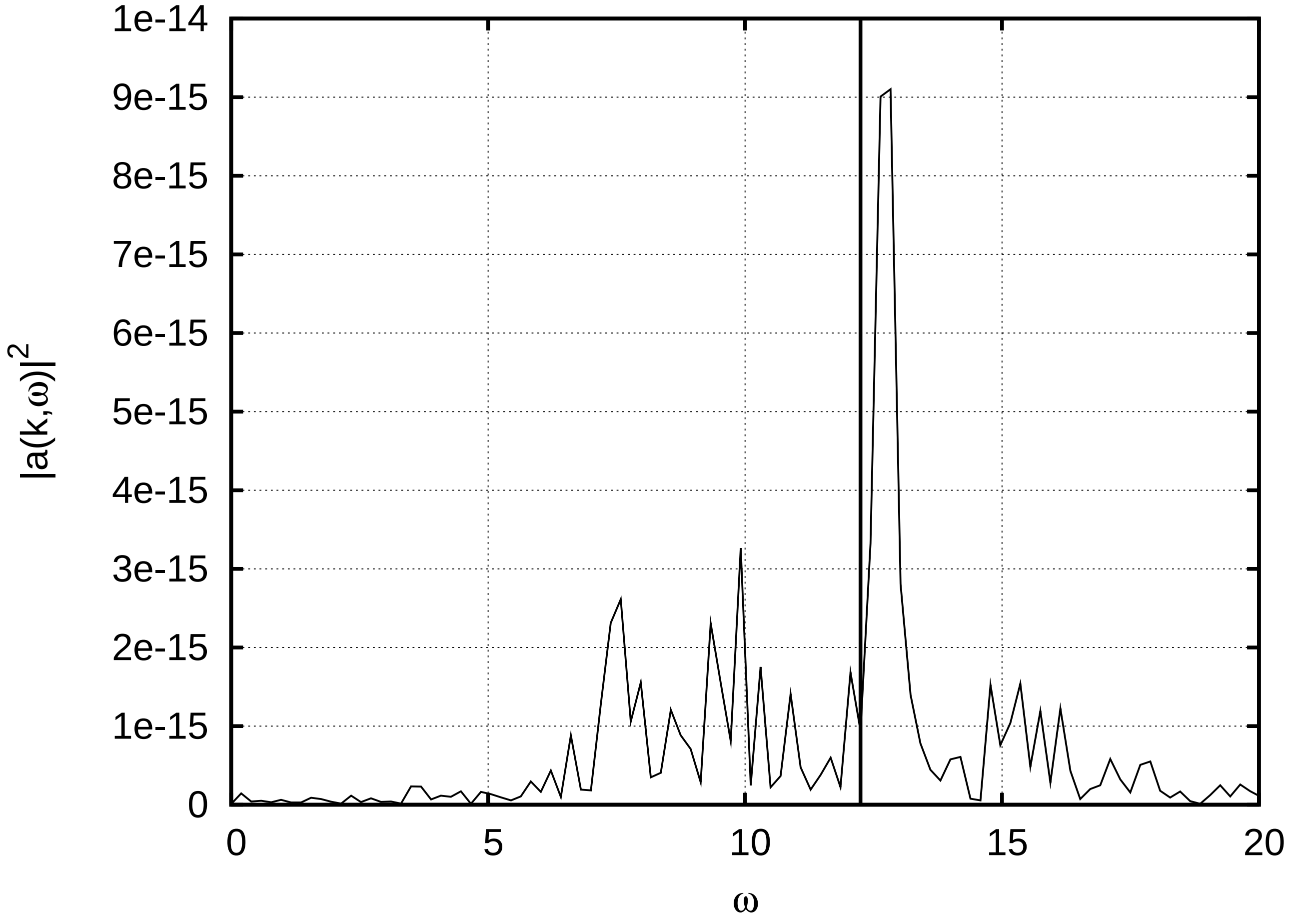
$n(\vec{k},\omega)$. With condensate. $\vec{k}=(0;50)$.



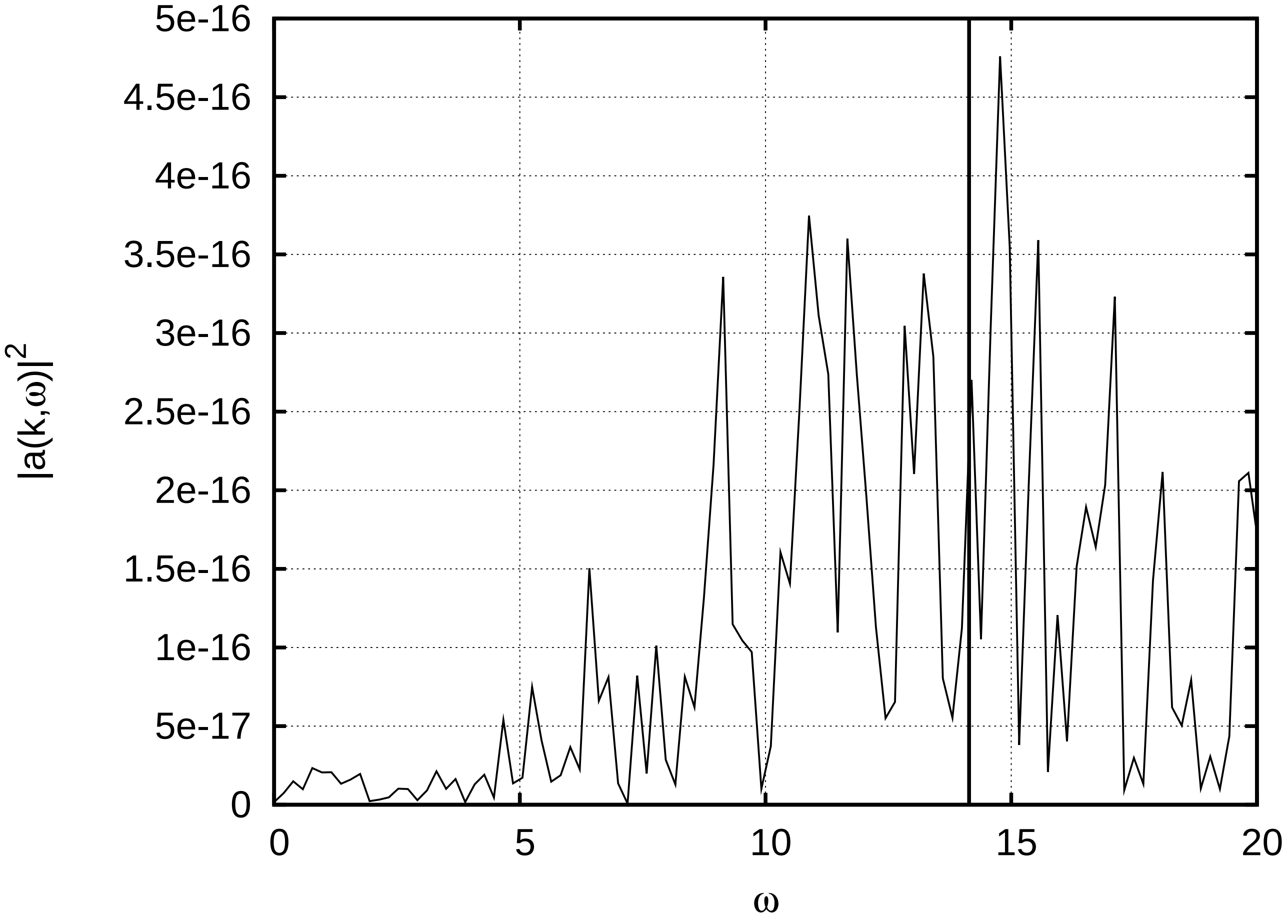
$n(\vec{k},\omega)$. With condensate. $\vec{k}=(0;100)$.



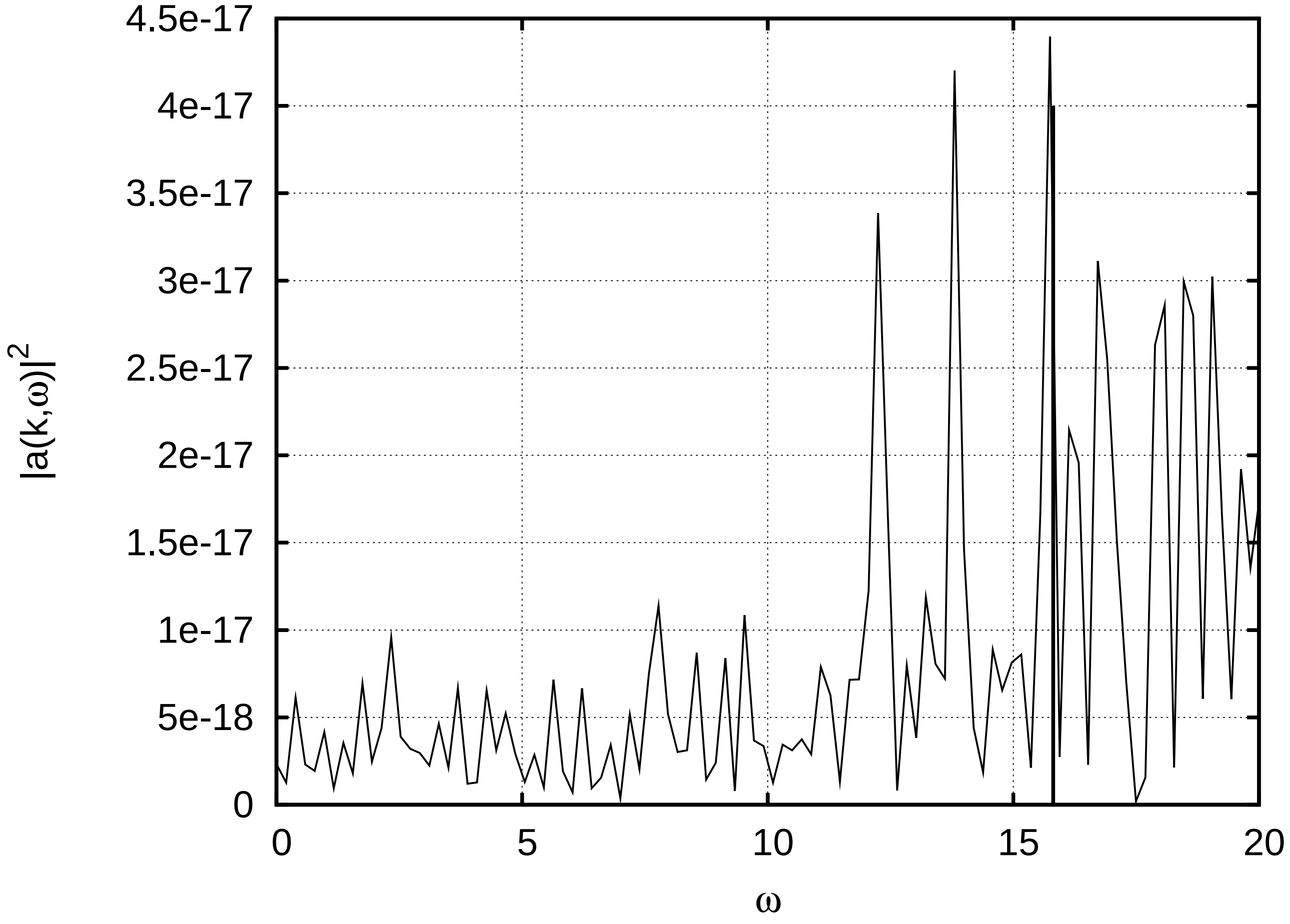
$n(\vec{k},\omega)$. With condensate. $\vec{k} = (0;150)$.



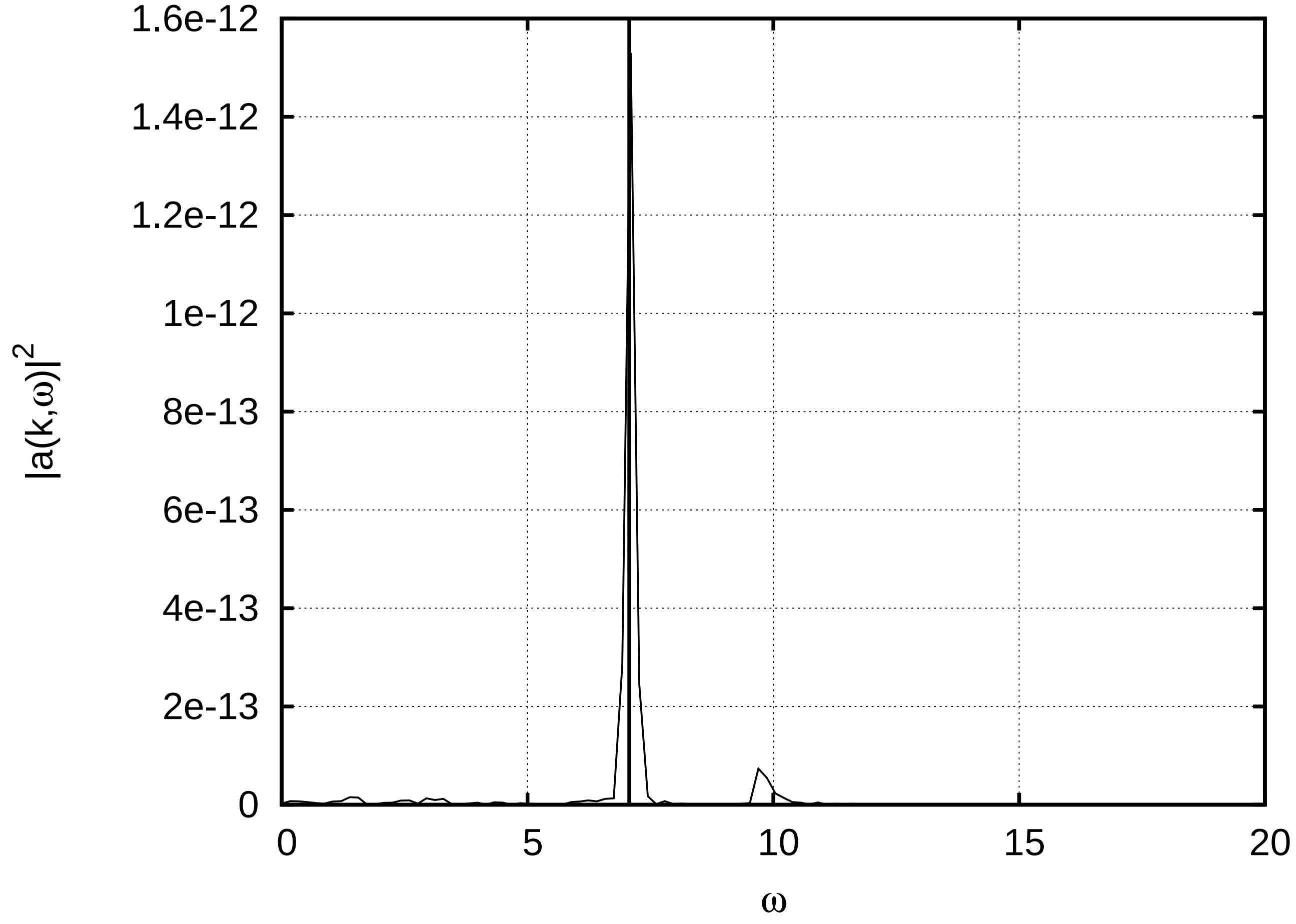
$n(\vec{k},\omega)$. With condensate. $\vec{k}=(0;200)$.



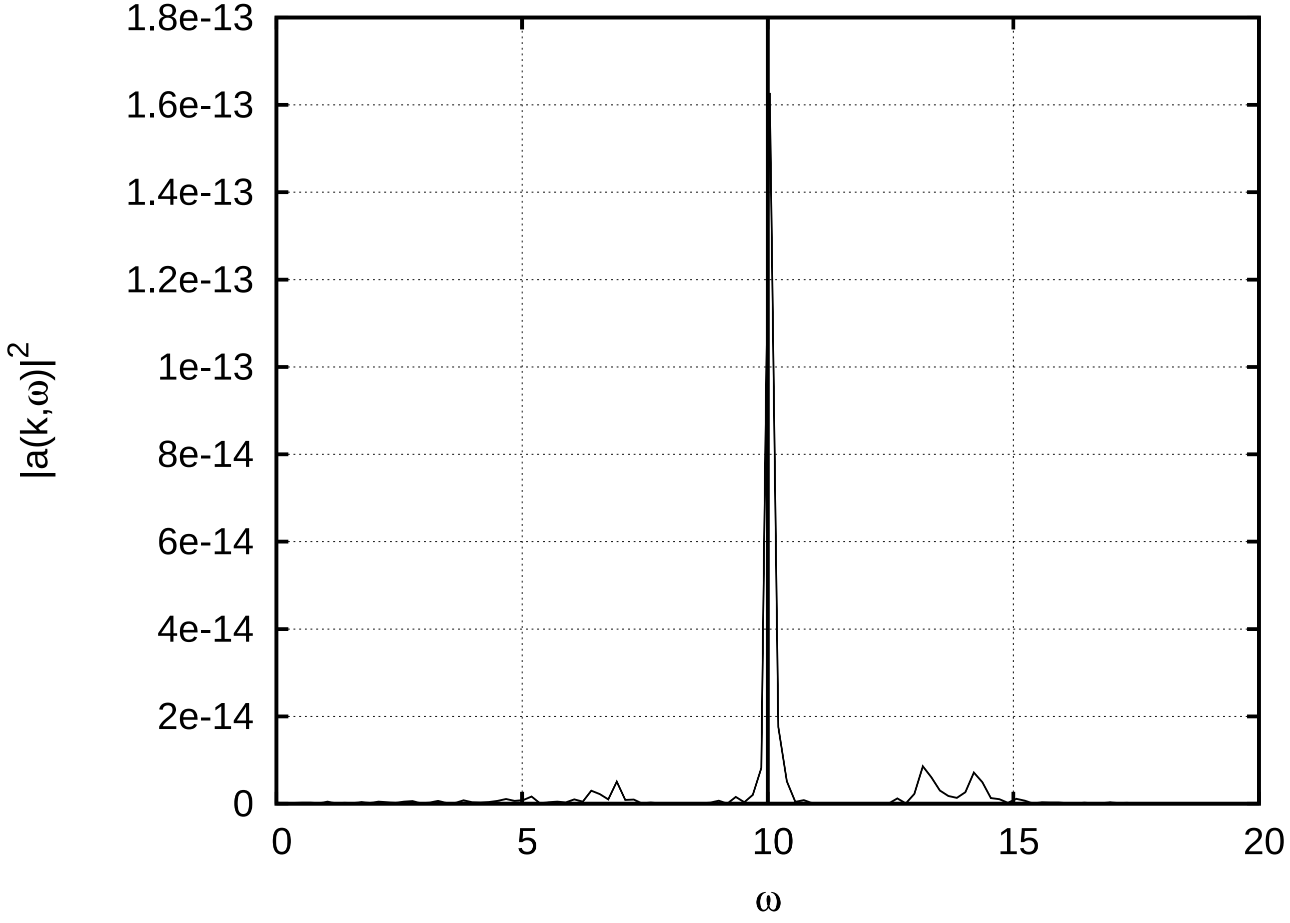
$n(\vec{k},\omega)$. With condensate. $\vec{k}=(0;250)$.



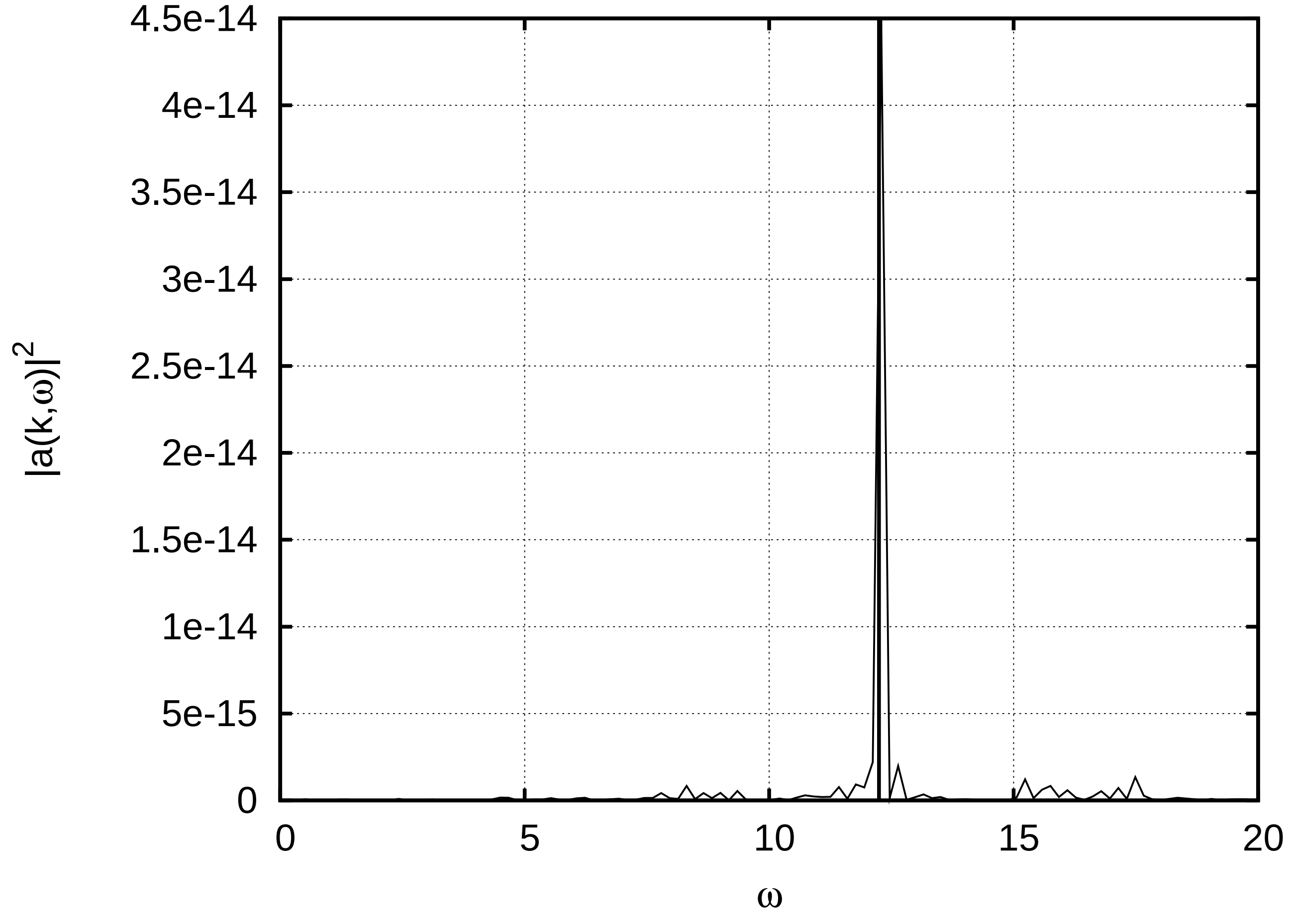
$n(\vec{k},\omega)$. Without condensate. $\vec{k} = (0;50)$.



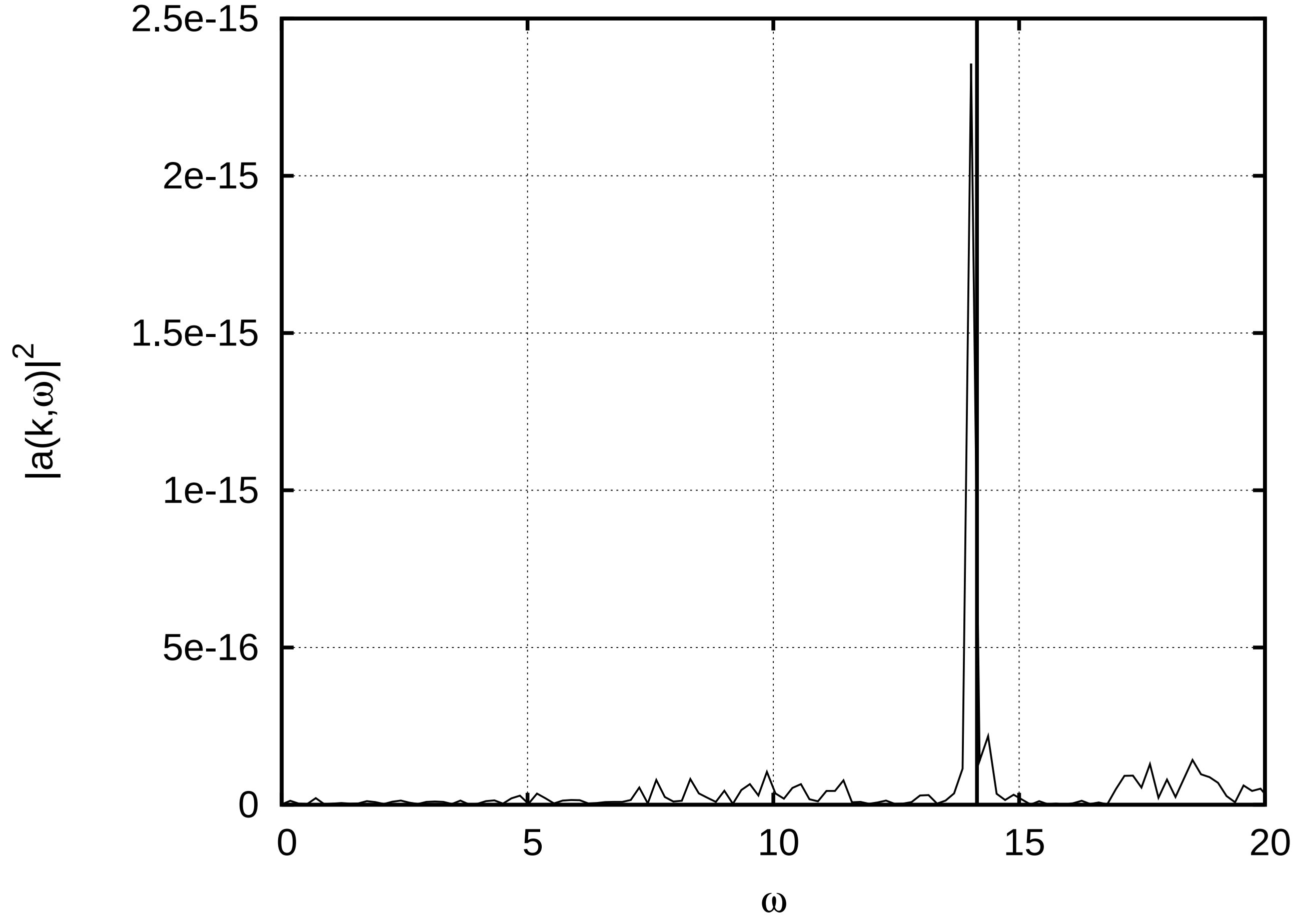
$n(\vec{k},\omega)$. Without condensate. $\vec{k}=(0;100)$.



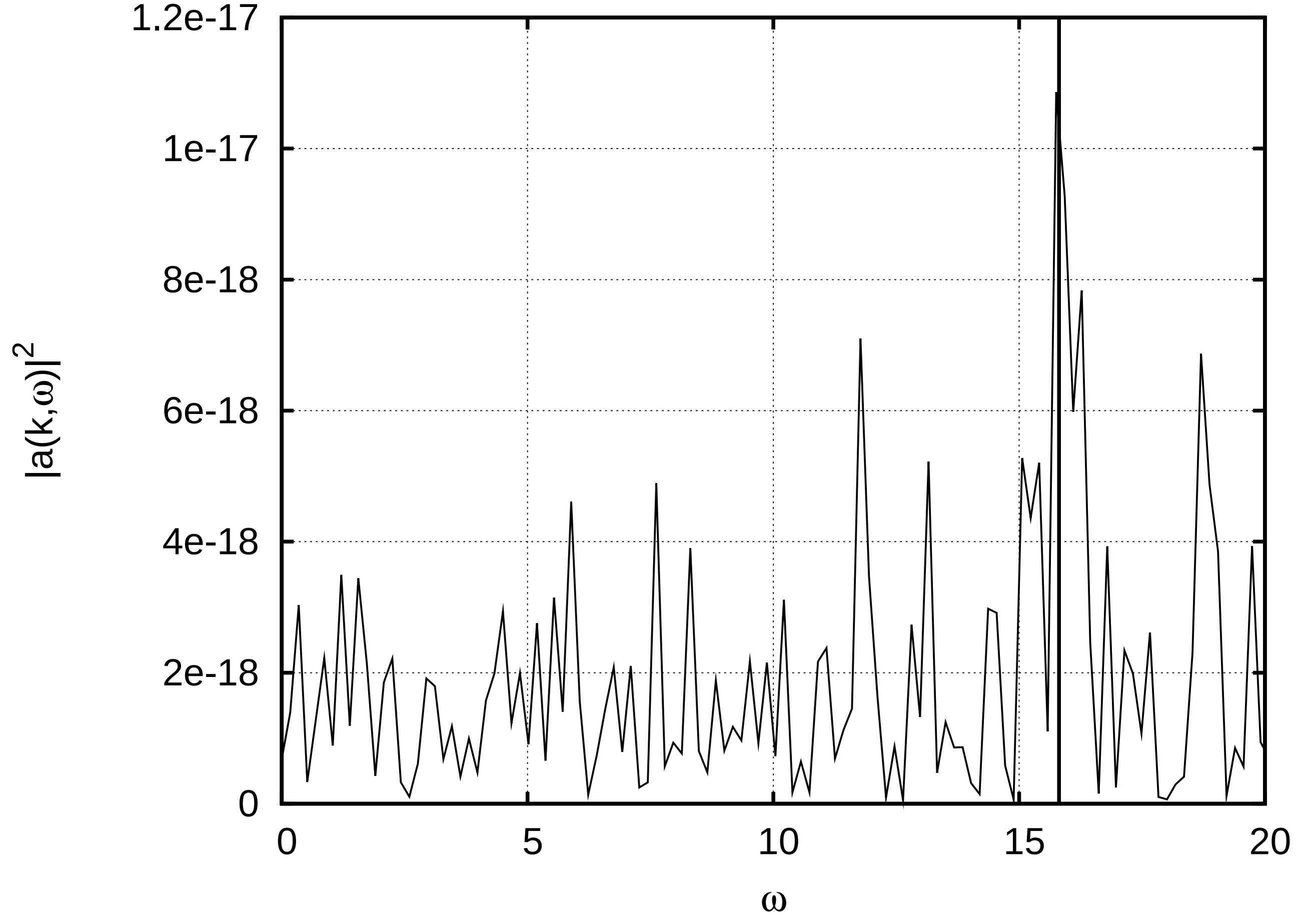
$n(\vec{k},\omega)$. Without condensate. $\vec{k}=(0;150)$.



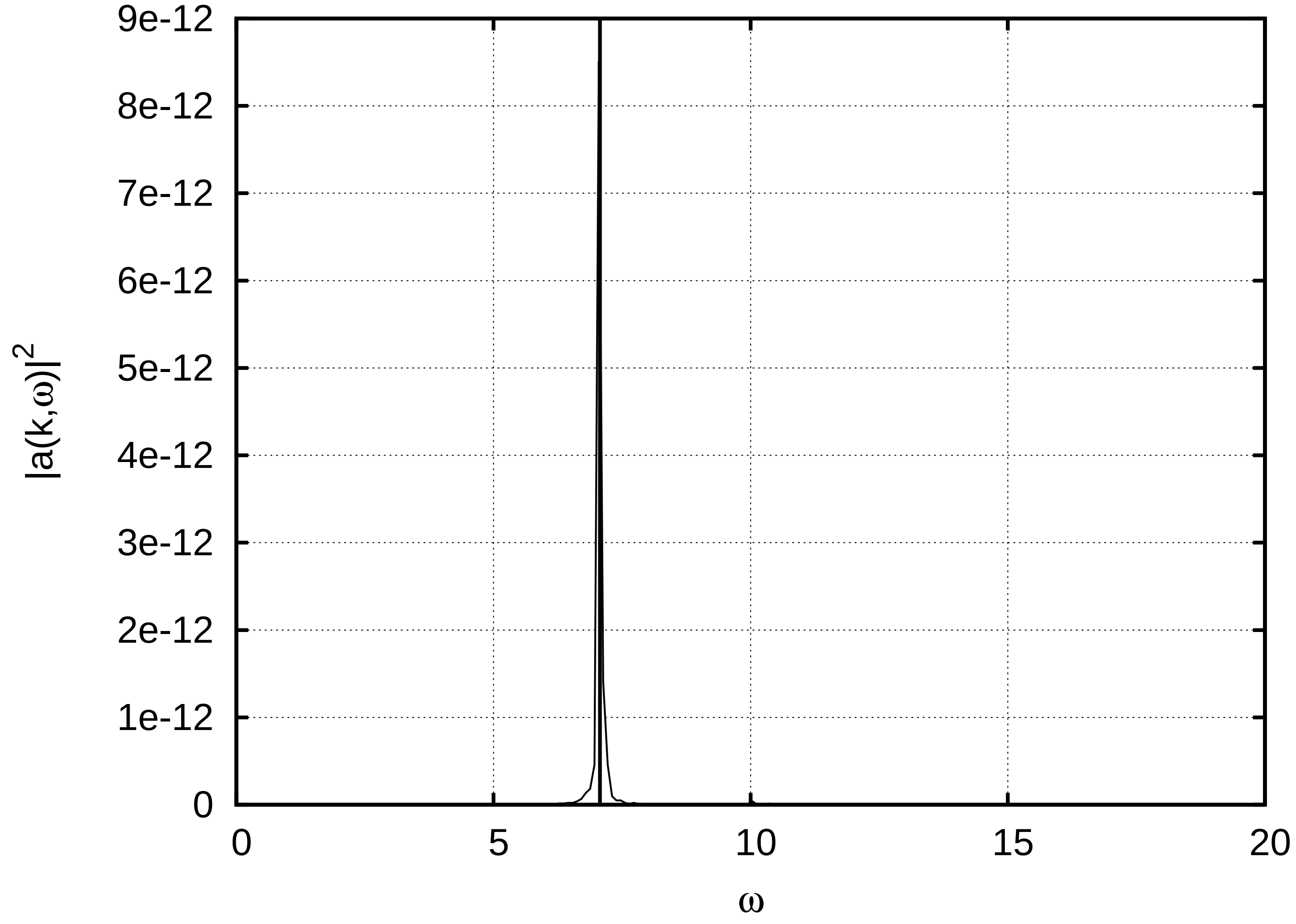
$n(\vec{k}, \omega)$. Without condensate. $\vec{k} = (0; 200)$.



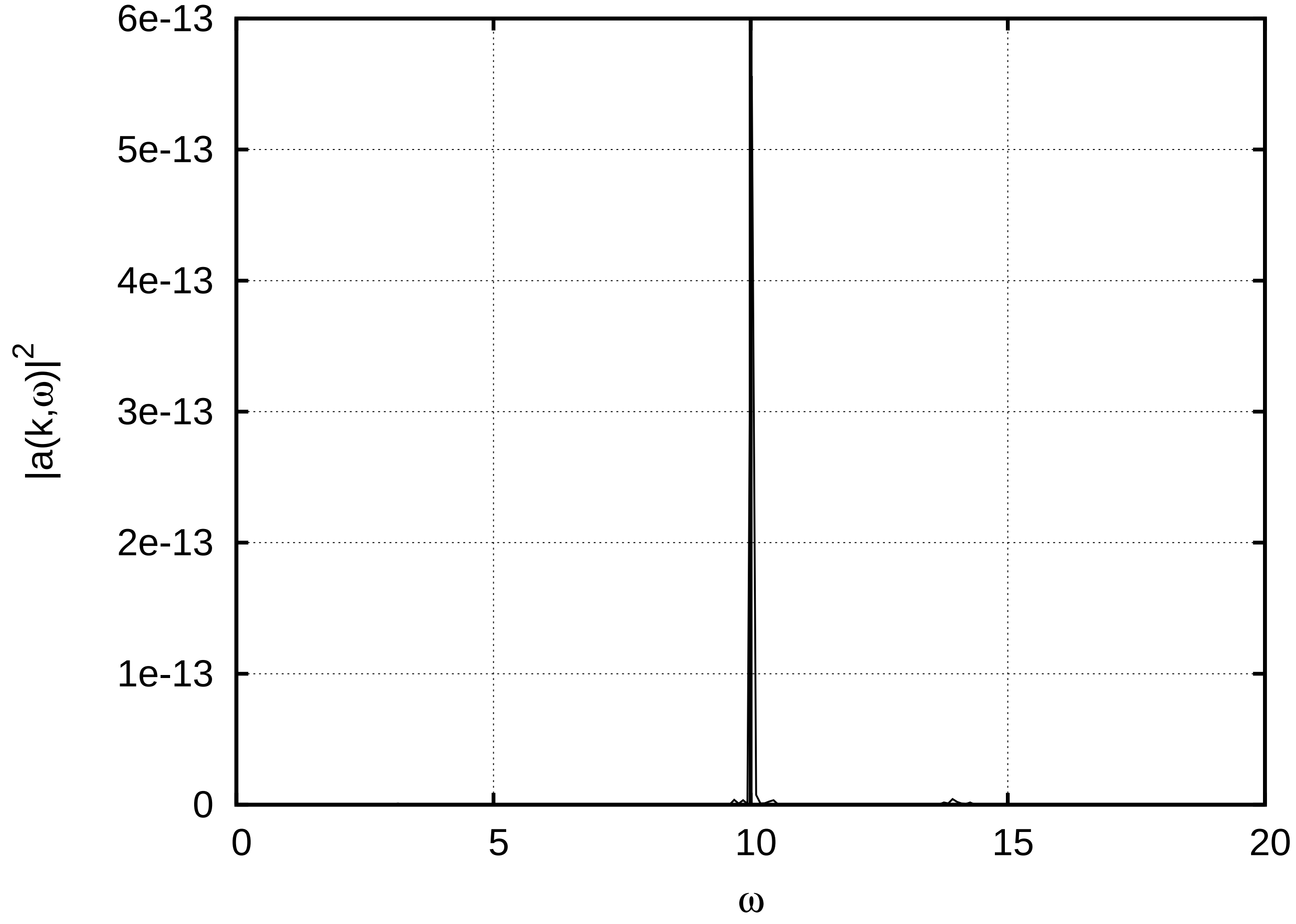
$n(\vec{k},\omega)$. Without condensate. $\vec{k}=(0;250)$.



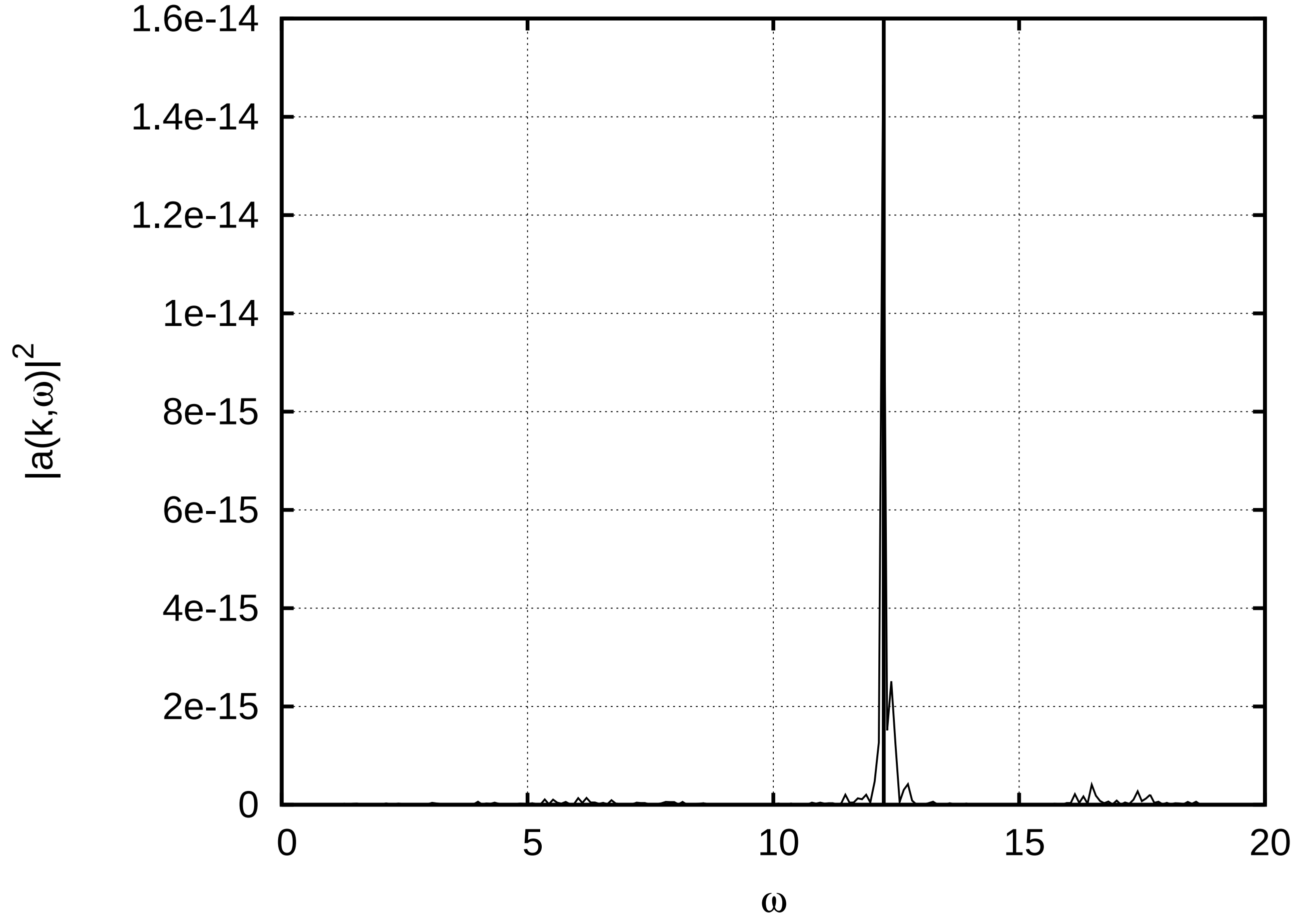
$n(\vec{k},\omega)$. Without anything. $\vec{k}=(0;50)$.



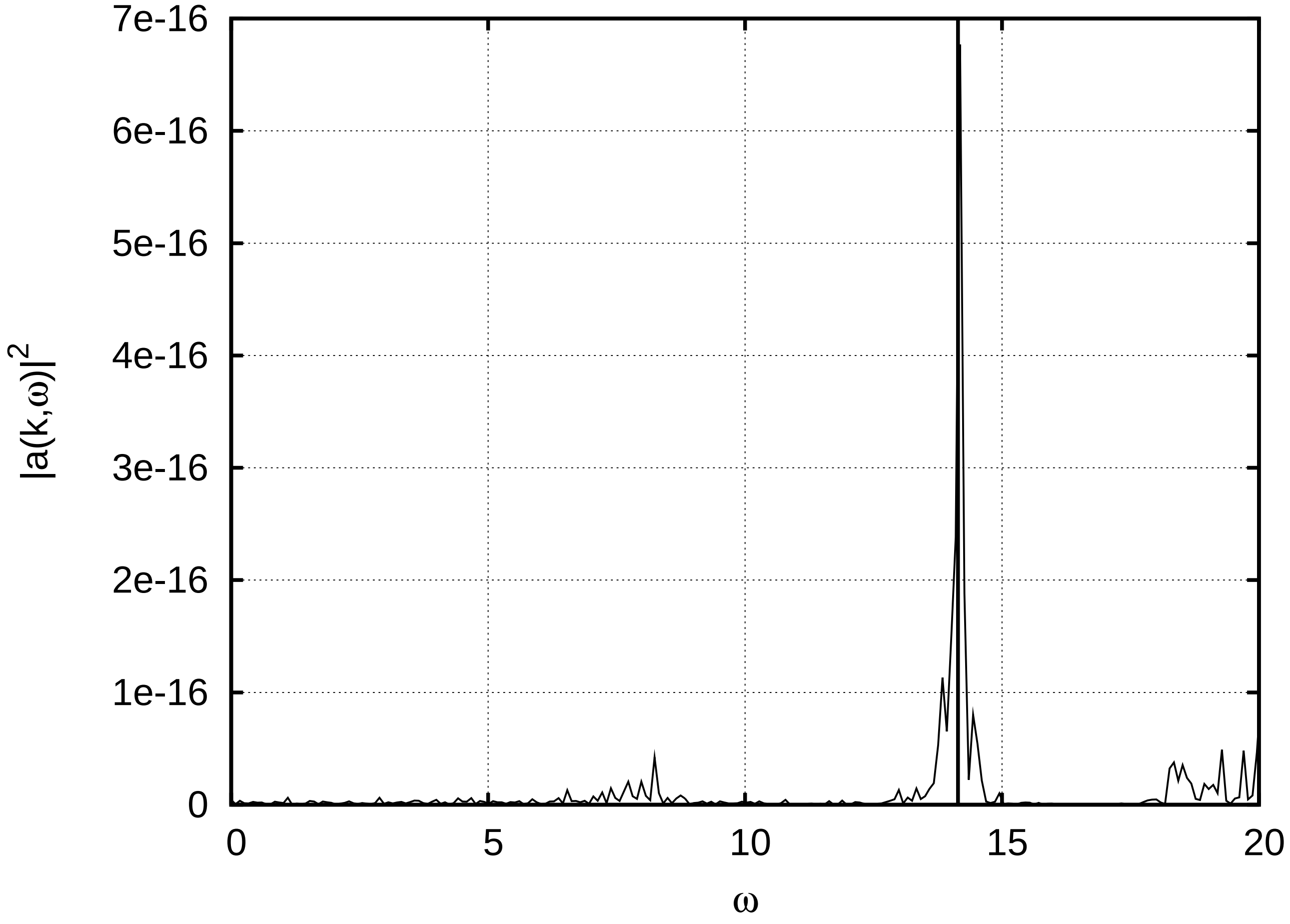
$n(\vec{k},\omega)$. **Without anything.** $\vec{k}=(0;100)$.



$n(\vec{k},\omega)$. **Without anything.** $\vec{k}=(0;150)$.



$n(\vec{k},\omega)$. **Without anything.** $\vec{k}=(0;200)$.



$n(\vec{k},\omega)$. **Without anything.** $\vec{k}=(0;250)$.