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Passive scalar transport in peripheral regions of random flows

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We designate the passive scalar field as θ. It can represent both, temperature variations or concentration of pollutants. The passive scalar evolution in an external flow is described by the equation

$$\partial_t\theta + \boldsymbol{v}\nabla\theta = \kappa\nabla^2\theta\,,$$

where $\boldsymbol{v}$ is the flow velocity and κ is the diffusion (thermodiffusion) coefficient. The coefficient is assumed to be small.

At investigating the passive scalar dynamics the velocity can be treated as short correlated in time and closed equations can be derived for the passive scalar correlation functions

$$F_n(t,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_n) = \langle\theta(t,\boldsymbol{r}_1)\ldots\theta(t,\boldsymbol{r}_n)\rangle\,,$$

obtained by averaging over times larger than the velocity correlation time.

One can derive closed equations

$$\partial_t F_n = \kappa \sum_{m=1}^{n} \nabla_m^2 F_n$$

$$+ \sum_{m,k=1}^{n} \sum_{\alpha\beta} \partial_{m\alpha} \left[D_{\alpha\beta}(\boldsymbol{r}_m, \boldsymbol{r}_k) \partial_{k\beta} F_n \right] ,$$

where the object D is expressed via the pair velocity correlation function as

$$D_{\alpha\beta}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \int_0^\infty dt' \, \langle v_\alpha(t+t', \boldsymbol{r}_1) v_\beta(0, \boldsymbol{r}_2) \rangle \, .$$

A z-dependence of the eddy diffusion tensor components can be found directly from the proportionality laws $v_x, v_y \propto z$ and $v_z \propto z^2$. Say,

$$D_{zz}(x, y, z_1; x, y, z_2) = \mu z_1^2 z_2^2\,,$$

where μ is a constant characterizing strength of the velocity fluctuations in the peripheral region.

The equation for the first moment of θ is

$$\partial_t \langle \theta \rangle = \partial_z \left[\mu z^4 \partial_z \langle \theta \rangle \right] + \kappa \partial_z^2 \langle \theta \rangle \, ,$$

Comparing two terms in RHS, one finds a characteristic diffusion length

$$r_{bl} = (\kappa/\mu)^{1/4}.$$

The quantity determines the thickness of the diffusion boundary layer.

We are interested mainly in the passive scalar transport through the region $z \gg r_{bl}$, where the passive scalar is carrying from the diffusive boundary layer to bulk. There we arrive at the proportionality law

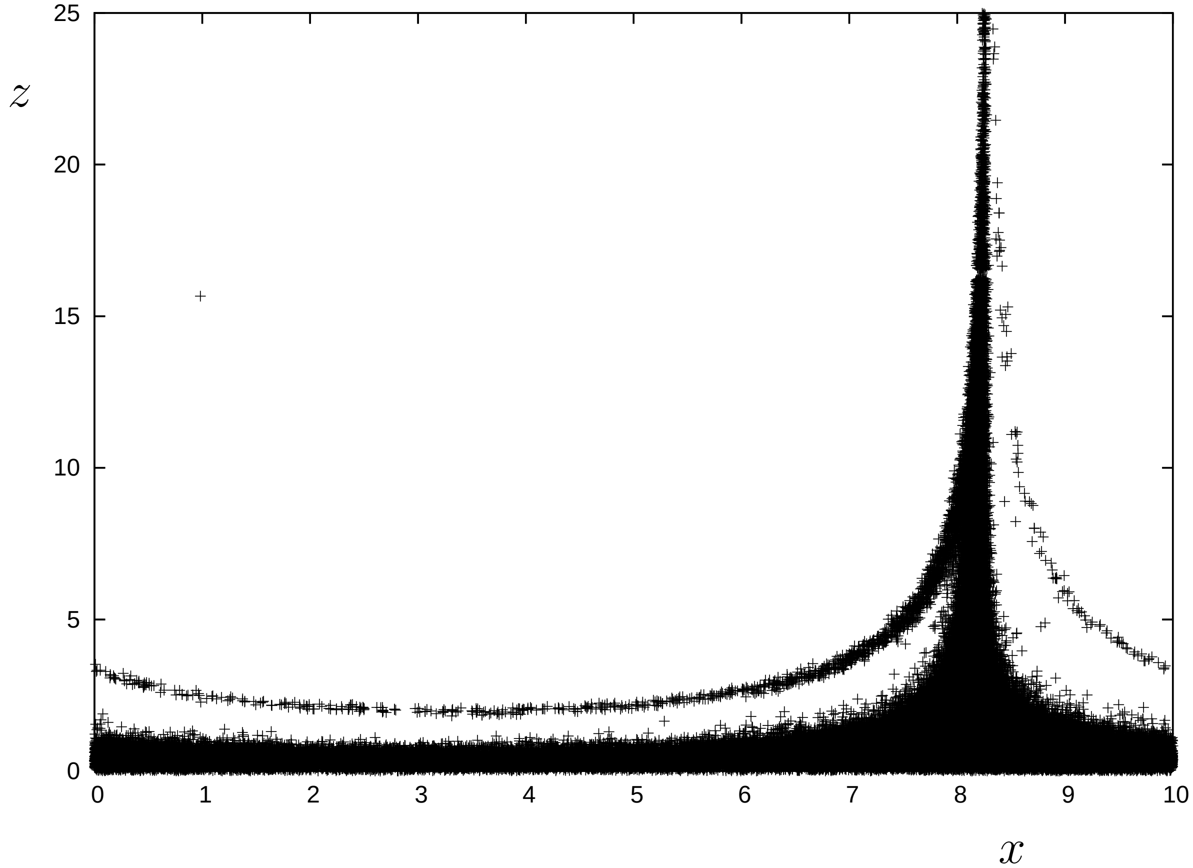
$$\langle\theta\rangle \propto z^{-3},$$

that gives the decaying rate of the average θ as z grows.

We introduce scaling exponents for the high passive scalar moments as well

$$\langle\theta^n\rangle \propto z^{-\eta_n},$$

at $z \gg r_{bl}$. Would the molecular diffusion be irrelevant there then $\eta_n = 3$. Really, the diffusion is relevant and values of the exponents η_n are subject of a special investigation.



z
25
20
15
10
5
0
0
1
2
3
4
5
6
7
8
9
10
x

One can define the passive scalar correlation length l (along the wall), that can be found by balance of the molecular and the eddy diffusion along the wall:

$$l \sim \sqrt{\kappa/\mu}\ z^{-1}.$$

The quantity is of order of r_{bl} at $z \sim r_{bl}$ and diminishes as z grows.

To exclude the effect of the molecular diffusion, we introduce an integral of the passive scalar field

$$\Theta(t,z) = A^{-1}\int dx\ dy\ \theta(t,x,y,z)\,,$$

where A is the area of the surface and z is its separation from the wall. Obviously $\langle\Theta\rangle \propto z^{-3}$. What about high-order moments?

Assuming that the passive scalar correlation length is smaller than the velocity one, we can derive

$$\partial_t \Phi_n(t, z_1, \ldots, z_n) = \mu \sum_{m,k=1}^{n} \frac{\partial}{\partial z_m} \left(z_m^2 z_k^2 \frac{\partial}{\partial z_k} \Phi_n \right)$$
$$+ 2\mu \sum_{m \neq k} \frac{\partial}{\partial z_m} \left(z_m^2 z_k \Phi_n \right) ,$$
$$\Phi_n(t, z_1, \ldots, z_n) = \langle \Theta(t, z_1) \ldots \Theta(t, z_n) \rangle .$$

The equation leads to the following closed equation for the moments of the integral passive scalar

$$\partial_t \langle \Theta^n \rangle = \mu \left[z^4 \partial_z^2 + 4nz^3 \partial_z + 4n(n-1)z^2 \right] \langle \Theta^n \rangle \,.$$

The equation leads to the scaling

$$\langle \Theta^n \rangle \propto z^{-\zeta_n}, \qquad \zeta_n = 2n - 1/2 + \sqrt{2n + 1/4}\,.$$

A natural conjecture that enables one to relate the moments of θ and those of Θ is in using the correlation length l as a recalculation factor:

$$\langle \Theta^n \rangle \sim \frac{l^{(d-1)(n-1)}}{A^{n-1}} \langle \theta^n \rangle,$$

$$\eta_n = \zeta_n - (n-1)(d-1).$$

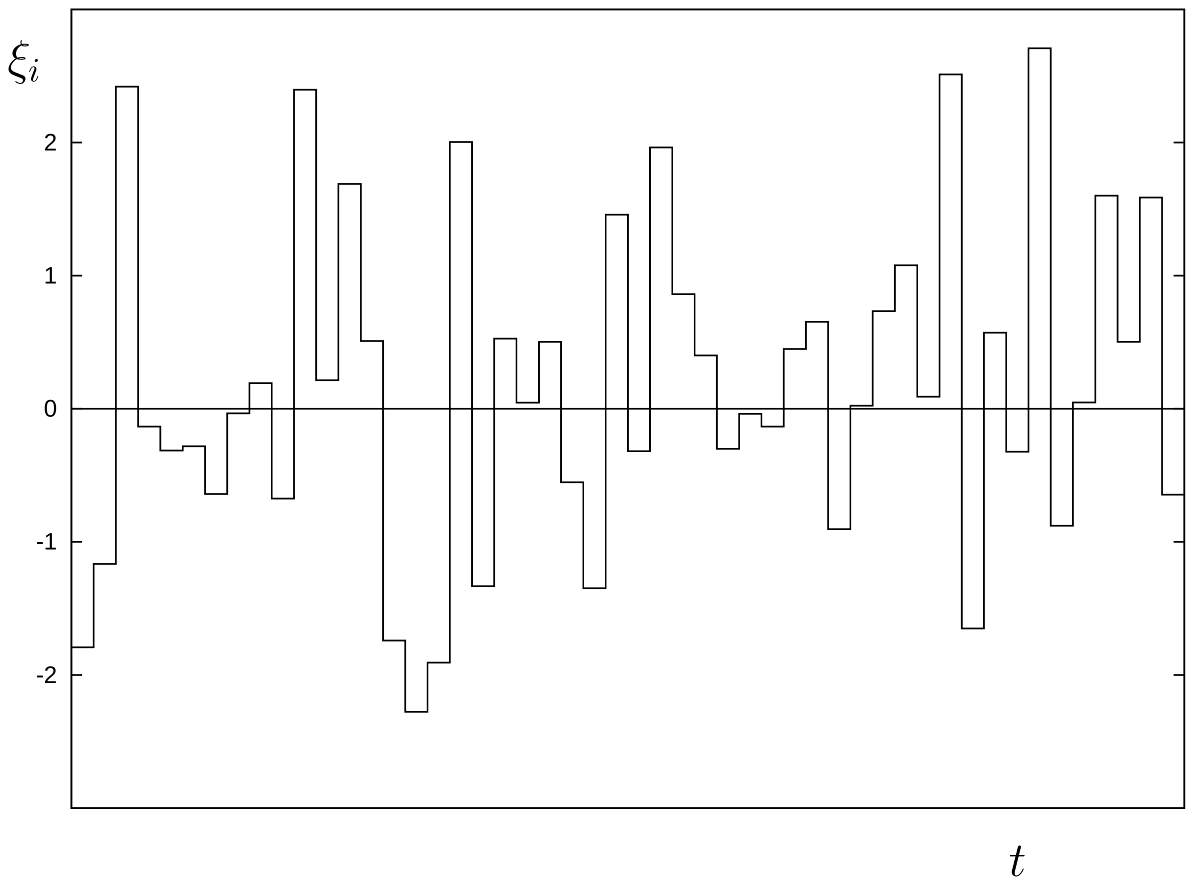
Here d is dimensionality of space.

We conducted Lagrangian simulations where dynamics of a large number of particles subjected to flow advection and Langevin forces (producing diffusion) is examined. The set of the particles is used instead of the passive scalar field θ, that can be treated as density of the particles. A big advantage of the approach is its applicability to a number of space dimensions d.

In our scheme a particle trajectory $\boldsymbol{\varrho}(t)$ obeys the equation

$$\partial_t \boldsymbol{\varrho} = \boldsymbol{v}(t, \boldsymbol{\varrho}) + \zeta(t),$$

where the first term represents the particle advection and the second term represents the Langevin force. The variables ζ are independent for different particles whereas the velocity is the same.


ξ_i
2
1
0
-1
-2
t

To establish principal qualitative features of the process, we perform mainly $2d$ simulations. The setup is periodic in x and the velocity in majority of runs was

$$v_x = z \left(\xi_1 \cos \frac{2\pi x}{L} + \xi_2 \sin \frac{2\pi x}{L} \right) \frac{L}{\pi},$$
$$v_z = z^2 \left(\xi_1 \sin \frac{2\pi x}{L} - \xi_2 \cos \frac{2\pi x}{L} \right),$$

where ξ_1 and ξ_2 are independent random functions of time. MOVIE

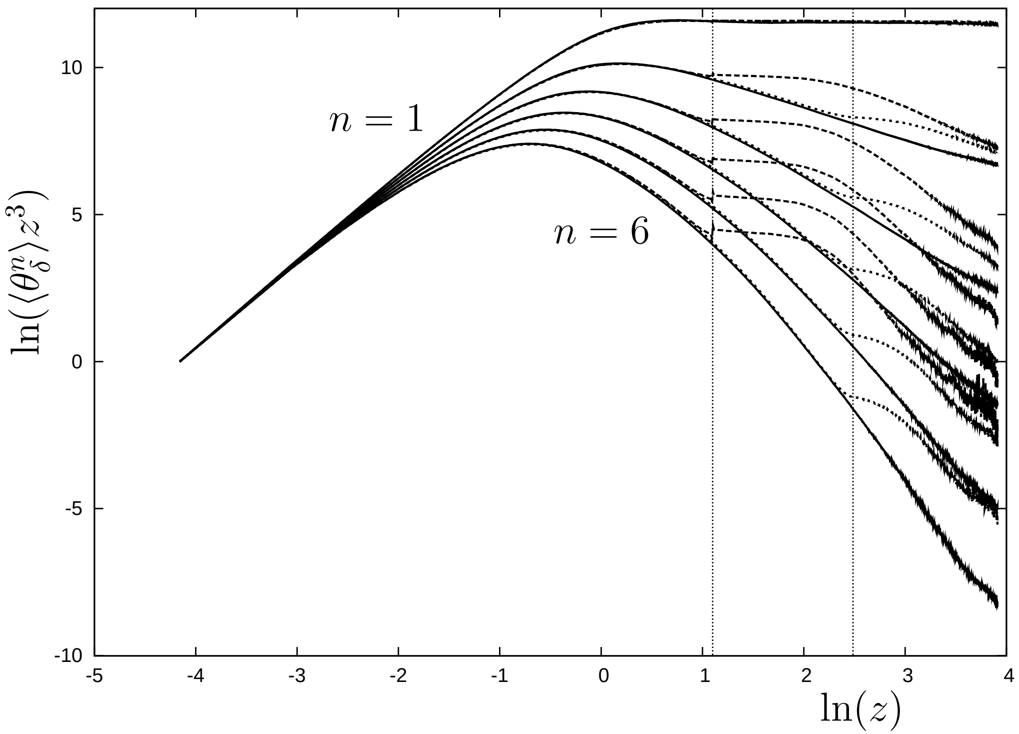


FIG. 1: Log-log plot of the moments of θ_δ, $\langle\theta_\delta^n\rangle$, multiplied by z^3, for $\delta = 0.03125$ and $n = 1 \div 6$. The graph reflects simulations where diffusion occurs everywhere, and is switched off at $z = 3$ or $z = 12$.

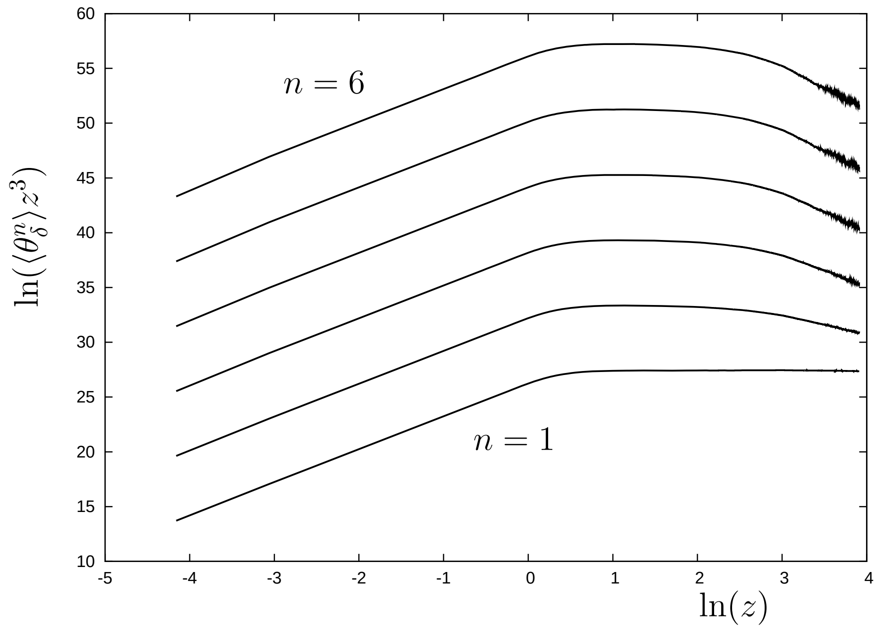


FIG. 1: Log-log plot of the moments of θ_δ, $\langle \theta_\delta^n \rangle$, multiplied by z^3, for $\delta = 0.03125$ and $n = 1 \div 6$ in the case where diffusion is substituted by a constant velocity carrying the particles from the wall.

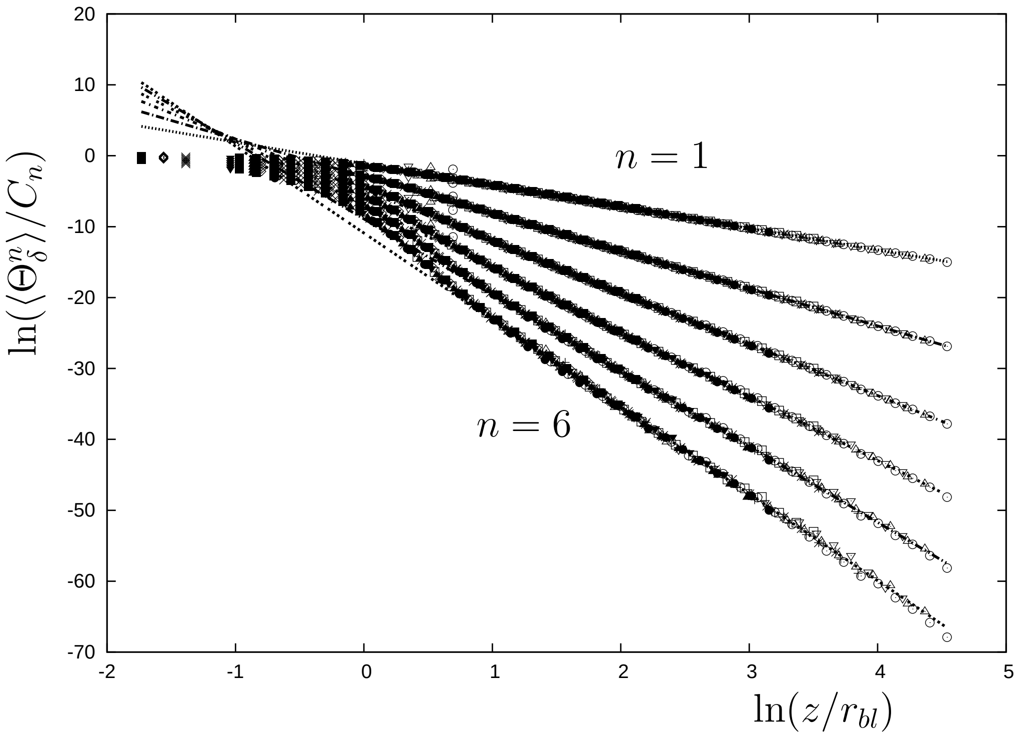


FIG. 1: Moments of Θ_δ in log-log coordinates, $n = 1 \div 6$. In the region $z > r_{bl}$ the results collapse onto single curves for three times $\tau = 0.001, 0.002, 0.004$ and four different values of the diffusion coefficient.

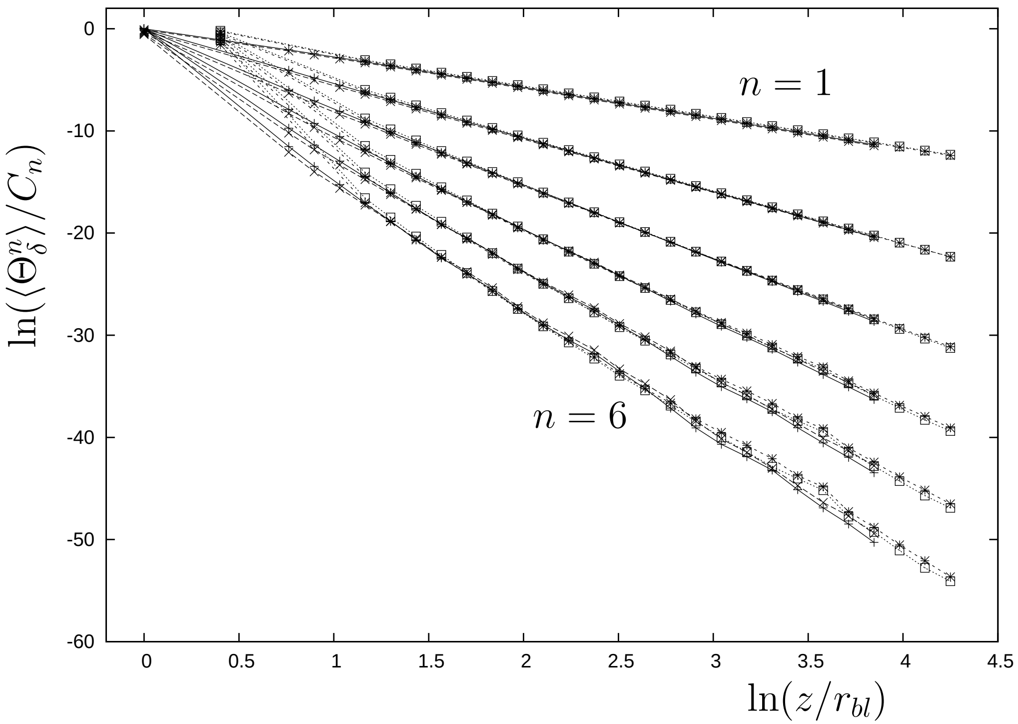


FIG. 1: Moments of Θ_δ in log-log coordinates, $n = 1 \div 6$. The results are obtained for two cases where the diffusion occurs everywhere and where it is switched off at $z > 3$, and also for two different velocity fields: with two and four harmonics.

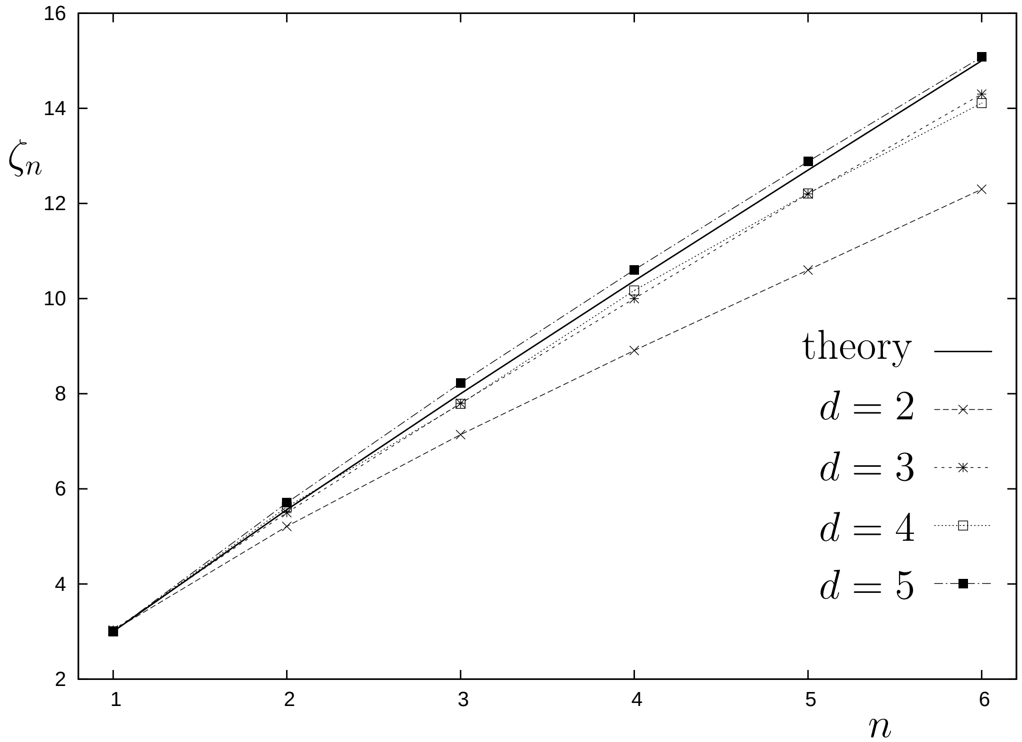


FIG. 1: Exponents of the moments $\langle \Theta_\delta^n \rangle$, for $n = 1 \div 6$ and space dimensions $d = 2 \div 5$. For comparison the theoretical curve $\zeta_n = 2n - 1/2 + \sqrt{2n + 1/4}$ is plotted (solid line).

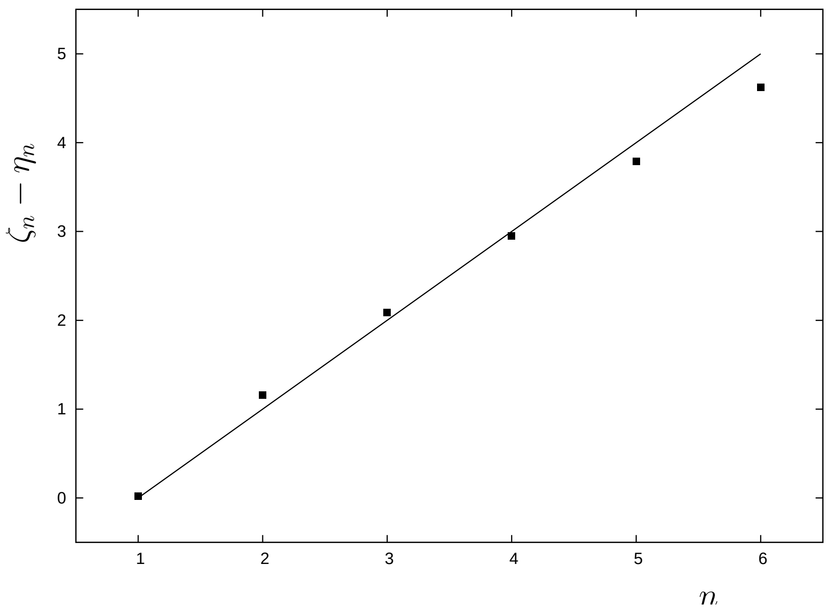


FIG. 1: The difference of the scaling exponents of moments for the integral passive scalar and for the passive scalar, $\zeta_n - \eta_n$, computed at $\delta = 0.03125$ in $2d$. For comparison, the theoretical prediction $n - 1$ is drawn.

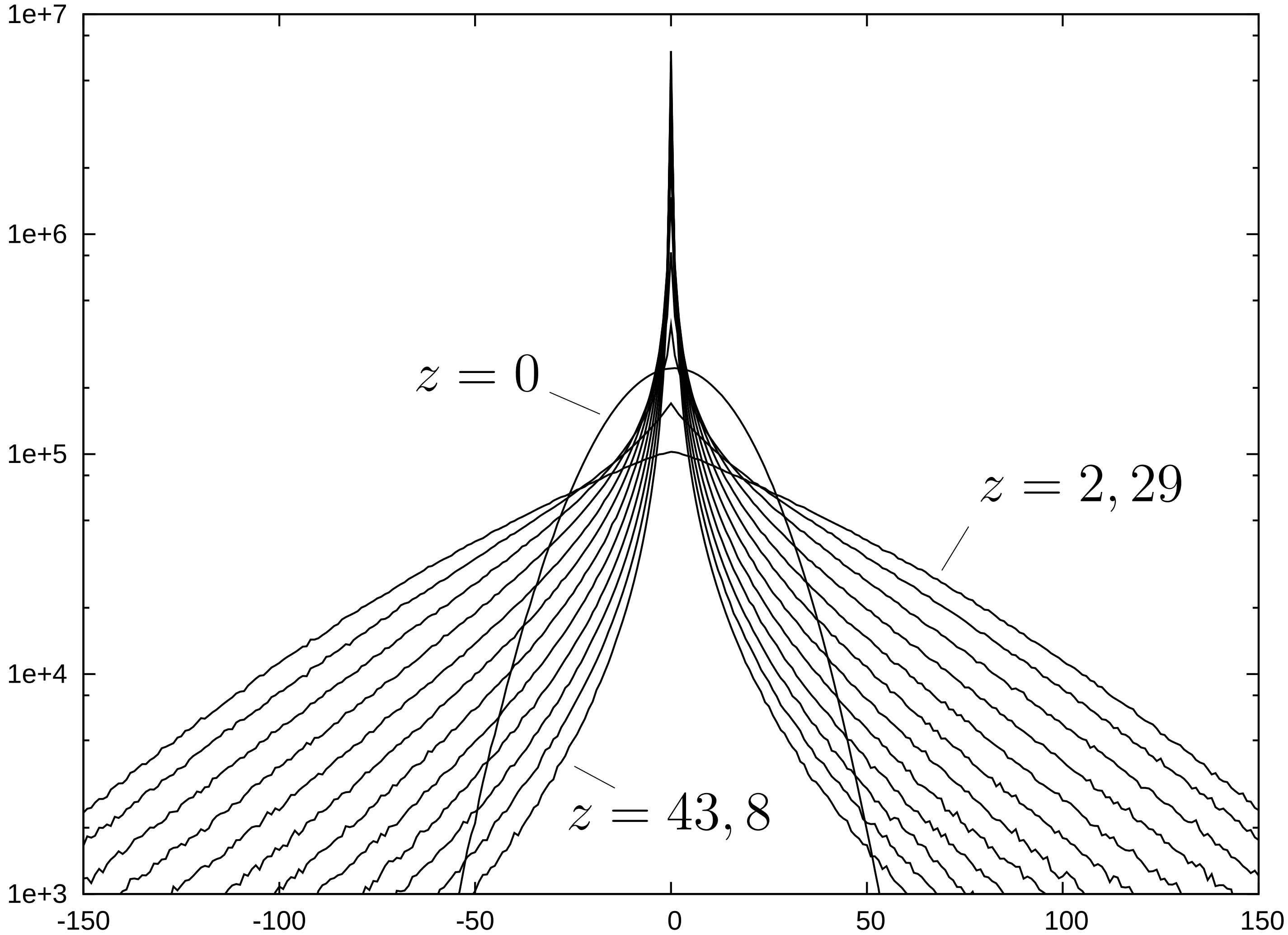


FIG. 1: Histograms of the passive scalar flux at different separations from the wall. The root mean square fluctuations are much larger than the average value and the histograms are practically symmetric. At $z = 0$ the probability distribution is Gaussian whereas at $z > r_{bl}$ it has exponential tails.

A comparison with numerics reveal deviations of the scaling exponents from the analytical predictions. Probably, the deviations are related to an existence of a long correlation along the wall that can be produced by the multi-fold structures. That should lead to increasing moments in comparison with the short correlated case. The conclusion is confirmed by numerics giving values of the scaling exponents for the moments of Θ that are smaller than the theoretical values in $2d$. The deviations diminish as d grows. Besides, the scaling of the short correlation length is in a good agreement with the theory.