Collapse and stable self-trapping for Bose-Einstein condensates with *$1/r^b$* type attractive interatomic interaction potential

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Catastrophic self-focusing of laser beam or collapse in Bose-Eistein condensate

Nonlinear Schrödinger equation:

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0$$

Catastrophic collapse (self-focusing):

$$\langle r^2 \rangle \to 0, \quad max\,|\psi| \to \infty$$

Temporal and spatial dependence of solutions near collapse: strong collapse vs. weak collapse

Strong collapse: Traps inco collapsing region a finite number of particles. Collapsing self-similar solution often has a form of rescaled ground-state soliton. Critical number of particles N_c is determined by the ground state soliton. For $N<N_c$ collpse is impossible

Weak collapse: Traps inco collapsing region a vanishing number of particles (close to a collapse time lesser and lesser particles are trapped into collapse). Critical number of particles is not defined (without trap).

The time-dependent Gross-Pitaevskii equation:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left\{ -\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{2} m \omega_0^2 (x_1^2 + x_2^2 + \gamma^2 x_3^2) + g|\Psi|^2 + \int V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r}' \right\} \Psi$$

$V(\mathbf{r})$ - long range interaction potential

$g = 4\pi\hbar^2 a/m,$

a - **s**-wave scattering length

Particular cases of $V(\mathbf{r})$

Dipole-dipole interaction potential:

$$V(\mathbf{r}-\mathbf{r}')=\frac{[\mathbf{d}_1(\mathbf{r})\cdot\mathbf{d}_2(\mathbf{r}')]-3[\mathbf{d}_1(\mathbf{r})\cdot\mathbf{u}][\mathbf{d}_2(\mathbf{r}')\cdot\mathbf{u}]}{|\mathbf{r}-\mathbf{r}'|^3},$$

$$\mathbf{u}=(\mathbf{r}-\mathbf{r}')/|\mathbf{r}-\mathbf{r}'|,$$

If $\mathbf{d}(\mathbf{r}) = const \quad \Rightarrow \quad V(\mathbf{r}) = \dfrac{1-3\cos^2\theta}{r^3}$

Gravity-like potential[1]: $V(\mathbf{r}) = -\dfrac{c}{r}$

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[1]O'Dell, Giovanazzi, Kurizki, and Akulin, Phys. Rev. Lett. **84**, 5687 (2000).

Consider a general potential of the following form

$$V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}, \quad b > 0, \quad \mathbf{n} \equiv \frac{\mathbf{r}}{r}, \quad r \equiv |\mathbf{r}|$$

$|f(\mathbf{n})| < \infty$ - arbitrary function of angle

Energy functional: $$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\delta E}{\delta\Psi^*}$$

$$E = E_K + E_P + E_{NL} + E_R,$$

Where $\frac{d\,E}{d\,t} = 0,$ and

$$E_K = \int \frac{\hbar^2}{2m}|\nabla\Psi|^2 d^3\mathbf{r},$$

$$E_P = \int \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2 + \gamma^2 x_3^2)|\Psi|^2 d^3\mathbf{r},$$

$$E_{NL} = \frac{g}{2}\int |\Psi|^4 d^3\mathbf{r},$$

$$E_R = \frac{1}{2}\int |\Psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r}d^3\mathbf{r}'.$$

The mean-square radius of the wave function:

$$\langle r^2 \rangle \equiv \int r^2 |\Psi|^2 d^3\mathbf{r}/N,$$

$N = \int |\Psi|^2 d^3\mathbf{r}$ - the total number of atoms in condensate

First time derivative of the mean-square radius:

$$\partial_t \langle r^2 \rangle = \frac{\hbar}{2mN} \int 2ix_j (\Psi \partial_{x_j} \Psi^* - \Psi^* \partial_{x_j} \Psi) d^3\mathbf{r}$$

Second time derivative of the mean-square radius:

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN}\Big[8E_K - 8E_P + 12E_{NL}$$
$$-2\int |\Psi(\mathbf{r}|^2 |\Psi(\mathbf{r}'|^2 (x_j \partial_{x_j} + x'_j \partial_{x'_j}) V(\mathbf{r}-\mathbf{r}') d^3\mathbf{r}\Big].$$

For $E_P = 0,\ V(\mathbf{r}) \equiv 0,$ we recover the virial theorem for NLS with local interaction [Vlasov, Petrishchev, Talanov (1971); Zakharov (1972)]:

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN}[12E - 4E_K]$$

Property of interaction potential $V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}$:

$$(x_j \partial_{x_j} + x'_j \partial_{x'_j}) V(\mathbf{r} - \mathbf{r}') = -bV(\mathbf{r} - \mathbf{r}')$$

$$\Rightarrow$$

The virial theorem is reduced to:

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN} \Big[4bE + (8-4b)E_K - (4+2b) m\omega_0^2 N \langle r^2 \rangle \\ -(4+2b) m\omega_0^2 N (\gamma^2 - 1) \langle x_3^2 \rangle + (12-4b) E_{NL} \Big]$$

$E_{NL} = \frac{g}{2} \int |\Psi|^4 d^3\mathbf{r}$, E_K and $\langle x_3^2 \rangle$ are unknowns

Collapse of BEC:

$$\langle r^2 \rangle \to 0, \quad max\,|\psi| \to \infty$$

from

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN}\Big[4bE + (8-4b)E_K - (4+2b)m\omega_0^2 N \langle r^2 \rangle$$
$$-(4+2b)m\omega_0^2 N(\gamma^2 - 1)\langle x_3^2 \rangle + (12-4b)E_{NL}\Big]$$

The trap can be ignored near collapse: $\omega_0 = 0$

$\Rightarrow$ The virial theorem is reduced to

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN}\left[4bE + (8-4b)E_K + (12-4b)E_{NL}\right]$$

$b > 3$ is similar to finite range potential; requires cutoff at small distance

$2 \leq b \leq 3$ collapse is possible even for g=0

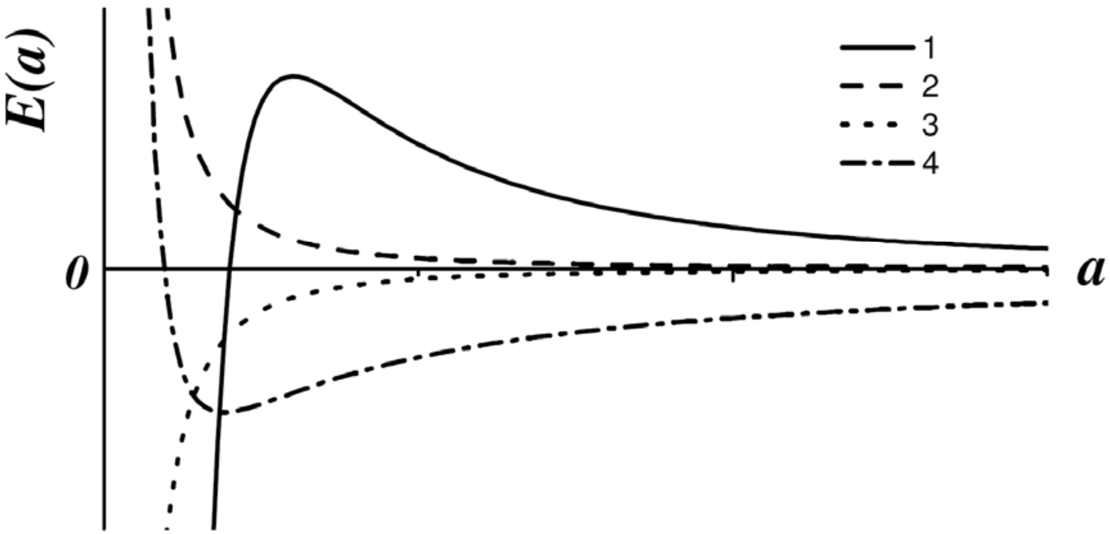
$b < 2$ collapse is impossible for g=0

P.M. Lushnikov, arxiv.org/1002.1469 (2010).

Scaling arguments (g=0, no trap)

Scaling transformation $\Psi(\mathbf{r}) \rightarrow a^{-3/2}\Psi(\mathbf{r}/a)$

$$E(a) = a^{-2}E_K + a^{-b}E_R$$



Curve 1: b>2

Curve 2: b=2 and $N<N_c$

Curve 3: b=2 and $N> N_c$

Curve 4: b<2

$$2 \leq b \leq 3$$

Assume for simplicity that contact interaction =0 (i. e. g=0)

$$\partial_t^2 \langle r^2 \rangle = \frac{1}{2mN}\left[4bE + (8-4b)E_K\right] \leq \frac{2bE}{mN}$$

$$\Rightarrow \quad \langle r^2 \rangle \leq \frac{bE}{mN} t^2 + \partial_t \langle r^2 \rangle|_{t=0} t + \langle r^2 \rangle|_{t=0}$$

If $\quad E < 0 \qquad \Rightarrow \qquad \langle r^2 \rangle \to 0, \quad max\,|\psi| \to \infty$

Sufficient condition for the collapse of BEC

We can do much better by the Kinetic Energy estimate:

$$\Psi \equiv R e^{i\phi}, \ R = |\Psi|$$

$$E_K = \frac{\hbar^2}{2m} \int [(\nabla R)^2 + (\nabla \phi)^2 R^2] d^3\mathbf{r},$$

Integrating by part and applying Cauchy-Schwartz inequality:

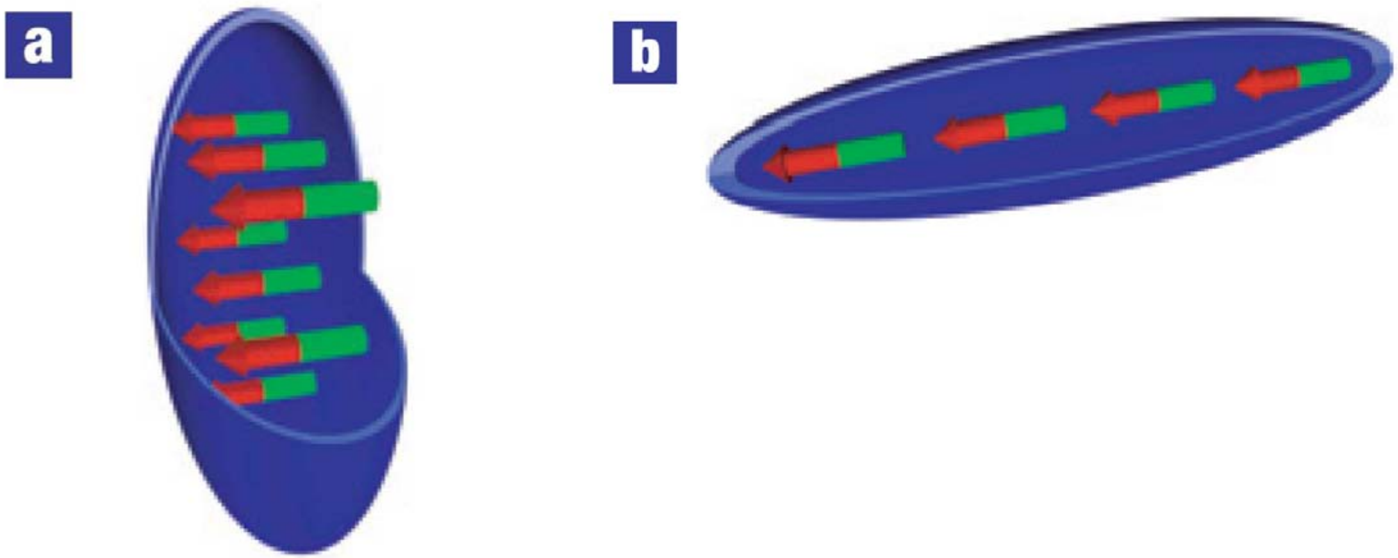
$$\frac{2mN}{\hbar} |\partial_t \langle r^2 \rangle| = 4 \left| \int x_j \partial_{x_j} \phi R^2 d^3\mathbf{r} \right|$$

$$\leqslant 4 \left(N \langle r^2 \rangle \int (\nabla \phi)^2 R^2 d^3\mathbf{r} \right)^{1/2},$$

$$N = -\frac{2}{3} \int x_j R \partial_{x_j} R d^3\mathbf{r} \leqslant \frac{2}{3} \left(N \langle r^2 \rangle \int (\nabla R)^2 d^3\mathbf{r} \right)^{1/2}$$

Particular case: dipolar Bose-Einstein condensate

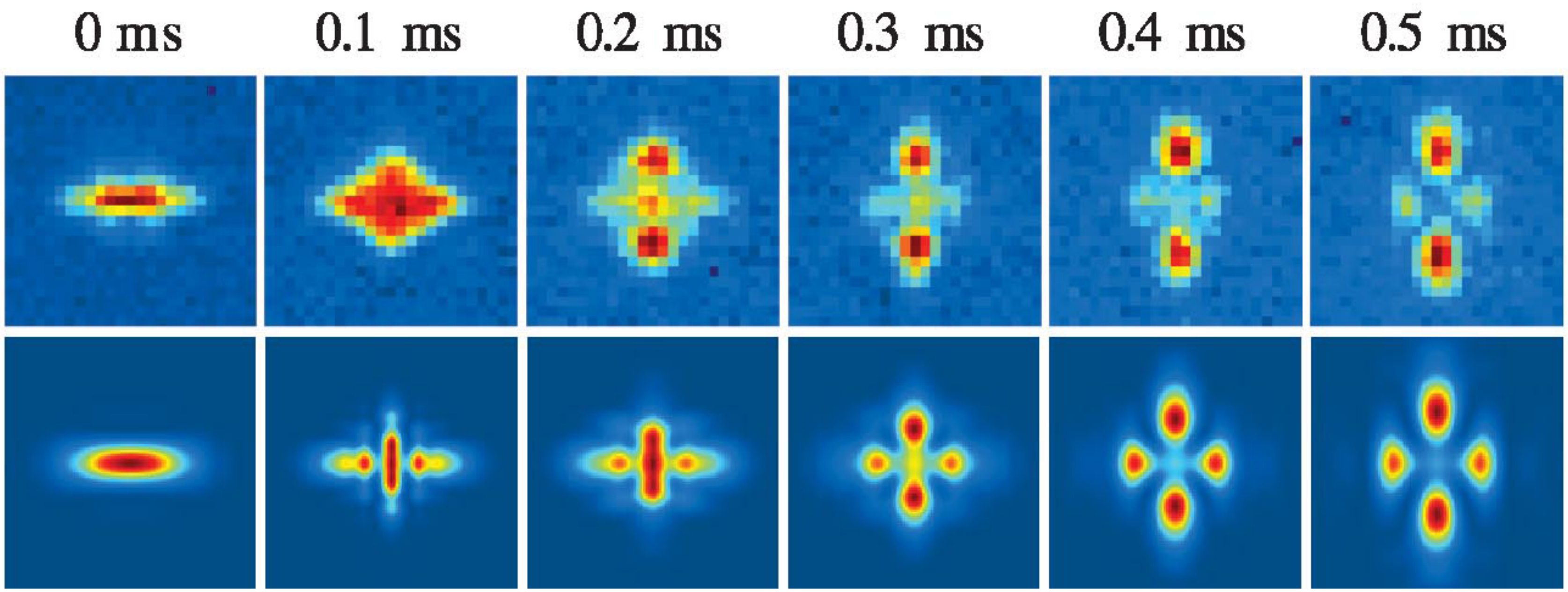
$$\mathbf{d}(\mathbf{r}) = const$$



a – stable condensate

b – collapse of condensate

Experimental observation of the dipolar Bose-Einstein condensate collapse



[1]T. Lahae et al. Phys. Rev. Lett. **101**, 080401 (2008).

$b < 2$ collapse is impossible for g=0

We use the inequality $\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^2} d^3\mathbf{r} \leq 4 \int |\nabla\Psi(\mathbf{r})|^2 d^3\mathbf{r}$

and generilize it using HÖlder' inequality

$$\int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r} = \int |\Psi(\mathbf{r})|^{2-b} \frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r}$$

$$\leq \left[\int \left(|\Psi(\mathbf{r})|^{2-b}\right)^{\frac{1}{1-b/2}} d^3\mathbf{r}\right]^{1-\frac{b}{2}} \left[\int \left(\frac{|\Psi(\mathbf{r})|^b}{|\mathbf{r}-\mathbf{r}'|^b}\right)^{\frac{2}{b}}\right]^{\frac{b}{2}} d^3\mathbf{r} \leq 2^b N^{1-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}}$$

But $V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}$ $\qquad f_m \equiv -\min_{\mathbf{n}} |f(\mathbf{n})|$

$\Rightarrow$

$\Rightarrow$

$$E_R = \frac{1}{2}\int |\Psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r} d^3\mathbf{r}' \geq -f_m\, 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}}$$

$\Rightarrow$ Energy is bounded from below (assuming g=0) by:

$$E \geq E_K - f_m\, 2^{b-1} N^{2-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}} \equiv P(E_K)$$

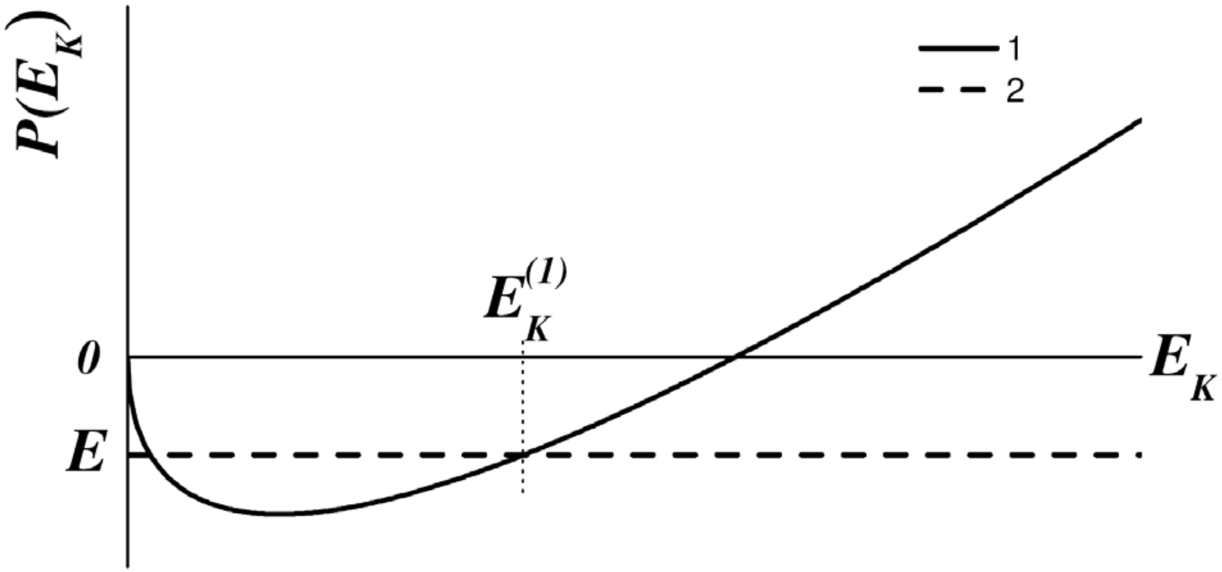
$$E = E_K + E_P + E_{NL} + E_R, \qquad E_K = \int \frac{\hbar^2}{2m} |\nabla\Psi|^2 d^3\mathbf{r},$$

$$E_P = \int \frac{1}{2} m\omega_0^2 (x_1^2 + x_2^2 + \gamma^2 x_3^2)|\Psi|^2 d^3\mathbf{r},$$

$$E_{NL} = \frac{g}{2}\int |\Psi|^4 d^3\mathbf{r},$$

$$E_R = \frac{1}{2}\int |\Psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r} d^3\mathbf{r}'.$$

$P(E_K)$ is bounded from below:



Curve 1: $P(E_K)$

$$\Rightarrow \quad E \geq -\frac{2-b}{b} 2^{-2} [f_m\, b]^{\frac{2}{2-b}} N^{\frac{4-b}{2-b}} \left(\frac{2m}{\hbar^2}\right)^{\frac{b}{2-b}}$$

E is bounded from below for any E_K

Collapse is impossible for $b < 2$ and g=0

Soliton solution

$$\Psi(\mathbf{r},t) = A(\mathbf{r})e^{-i\mu t/\hbar}$$

$$\Rightarrow \quad \left[-\mu - \frac{\hbar^2}{2m}\nabla^2 + \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2 + \gamma^2 x_3^2) + \int d^3\mathbf{r}'\, V(\mathbf{r}-\mathbf{r}')A(\mathbf{r}')^2\right]A(\mathbf{r}) = 0,$$

$$E_{K,s} = -\mu N_s \frac{b}{4-b} + E_{P,s}, \quad E_{R,s} = \mu N_s \frac{2}{4-b},$$

$$E_s = -\mu N_s \frac{b-2}{4-b} + 2E_{P,s},$$

where subsript "s" means that all integrals are taken for soliton solution.

For $\omega_0 = 0$ all integrals depends on N_s only.

Soliton solution is the stationary point of E for the fixed number of particles $\delta(E - \mu N) = 0$

As well as the stationary point of the following functional

$$\mathcal{F}(\Psi) \equiv N^{1-\frac{b}{2}} \left(\frac{2m}{\hbar^2} E_K\right)^{\frac{b}{2}} / \int \frac{|\Psi(\mathbf{r})|^2}{|\mathbf{r}-\mathbf{r}'|^b} d^3\mathbf{r}$$

Assume that $f(\mathbf{n}) = Const < 0$ then one can look At the radially-symmetric solution and obtain for the ground-state soliton solution that

$$\mathcal{F}(\Psi) \geq \min \mathcal{F}(\Psi) = \mathcal{F}(\Psi_{s,ground})$$

which gives stricter than previous inequality:

$$E \geq \min E = E_{s,ground}$$

I.e. the ground state soliton realizes a global minimum of E for fixed N.

$\Rightarrow$ the ground-state soliton is nonlinearly stable

Example: b=1 gravity-like potential. Ground state found in Ref[1]

[1]Cartarius, Fabčič, Main, and Wunner, Phys. Rev. A, **78**, 013615 (2008).

If $f(\mathbf{n}) \neq Const$ but takes negative values for nonzero range of values of ***n*** then ground state soliton still exist and stable.

If $f(\mathbf{n}) > 0$ all values of ***n*** then soliton solution does not exists (we assume g=0 and $\omega_0 = 0$) and Any initial conditions decays into linear waves. Self-trapping requires $\omega_0 \neq 0$ in that case.

Temporal and spatial dependence of solutions near collapse: strong collapse vs. weak collapse

Strong collapse: Traps inco collapsing region a finite number of particles. Collapsing self-similar solution often has a form of rescaled ground-state soliton. Critical number of particles N_c is determined by the ground state soliton. For $N<N_c$ collpse is impossible

Weak collapse: Traps inco collapsing region a vanishing number of particles (close to a collapse time lesser and lesser particles are trapped into collapse). Critical number of particles is not defined (without trap).

Nonlinear Schrödinger equation:

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0$$

D=2: Strong collapse

D=3: Both Weak and Strong collapses
are possible but Strong collapse is unstable

D=2

$$i\frac{\partial\psi(\mathbf{r})}{\partial t}+\nabla^2\psi(\mathbf{r})+|\psi(\mathbf{r})|^2\psi(\mathbf{r})=0$$

Scaling invariance of 2D Nonlinear Schrödinger Equation (NLS):

If $\Psi(\mathbf{r},t)$ - solution of NLS

$\Rightarrow \quad L^{-1}\Psi(\frac{\mathbf{r}}{L},\frac{t}{L^2})$ - also solution of the same NLS with $L=Const$

Critical case is at the border between collapsing and noncollapsing cases

For $V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^b}$ we have fixed dimension D=3 but instead we can change b.

b=2: critical case which has the scaling invariance.

If $\Psi(\mathbf{r}, t)$ - solution of GPE with $V(\mathbf{r}) = \frac{f(\mathbf{n})}{r^2}$

$\Rightarrow L^{-1}\Psi(\frac{\mathbf{r}}{L}, \frac{t}{L^2})$ - also solution of the same GPE with $L = Const$

$\Rightarrow$ For b=2 collapse is critical with self-similar solution determined by the ground-state soliton

E.g., for radially-symmetrical case $V(\mathbf{r}) = -\frac{c}{r^2}$
The self-similar collapsing solution of the GPE

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + \int \frac{|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|^2} d\mathbf{r}'\psi(\mathbf{r}) = 0$$

has the following form:

$$\Psi(\mathbf{r},t) = \Psi(r,t),$$

$$\Psi(r,t) \simeq \frac{1}{L}V(\rho)e^{i\tau+iLL_t\rho^2/4}, \quad L\to 0,$$

$$\rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')},$$

$$\nabla^2 V(\rho) - V(\rho) + \int \frac{|V(\rho')|^2}{|\rho-\rho'|^2} d\rho' V(\rho) = 0.$$

NLS:

$D<2$ – global existence of solution

$D=2$ – critical collapse

$D>2$ – supercritical collapse

GPE with $V(\mathbf{r}) = \dfrac{f(\mathbf{n})}{r^b}$

$b<2$ – global existence of solution

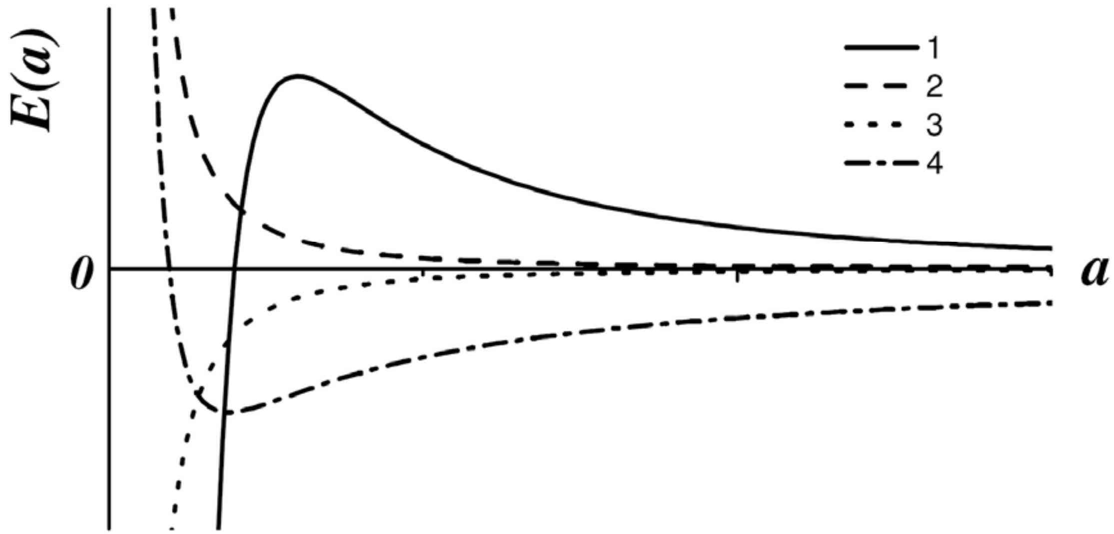
$b=2$ – critical collapse

$b>2$ – supercritical collapse

Scaling arguments (g=0, no trap)

Scaling transformation $\Psi(\mathbf{r}) \rightarrow a^{-3/2}\Psi(\mathbf{r}/a)$

$$E(a) = a^{-2}E_K + a^{-b}E_R$$



Curve 1: b>2

Curve 2: b=2 and N<N_c

Curve 3: b=2 and N> N_c

Curve 4: b<2

Nonlinear Schrödinger equation:

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0$$

D=2: Strong collapse

D=3: Both Weak and Strong collapses
are possible but Strong collapse is unstable

D=2

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) + |\psi(\mathbf{r})|^2\psi(\mathbf{r}) = 0$$

Scaling invariance of 2D Nonlinear Schrödinger Equation (NLS):

If $\Psi(\mathbf{r},t)$ - solution of NLS

$\Rightarrow \quad L^{-1}\Psi(\frac{\mathbf{r}}{L}, \frac{t}{L^2})$ - also solution of the same NLS with $L = Const$

Critical collapse: Self-similar solution near singularity

$$\Psi(\mathbf{x},t) = \Psi(r,t), \quad r = (x^2+y^2)^{1/2}$$

$$\Psi(r,t) \simeq \frac{1}{L} V(\rho) e^{i\tau + iLL_t\rho^2/4}, \quad L \to 0,$$

$$\rho = \frac{r}{L}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')},$$

Soliton solution of NLS: $\triangle V - V + |V|^2 V = 0$

LogLog law[1]: $L = \left(2\pi \frac{t_0 - t}{\ln\ln[1/(t_0-t)]} \right)^{1/2}.$

[1] G. Fraiman (1985), M. Landman, G. Papanicolaou, C. Sulem, and P. Sulem (1987).

NLS D=3: Supecritical case

Self-similar solution for weak collapse[1]

$$\psi = \frac{1}{(t_0 - t)^{1/2+i\alpha}} \chi\left(\frac{r}{(t_0 - t)^{1/2}}\right), \qquad \alpha \simeq 0.545\ldots$$

$$\chi(\xi) = \frac{c}{\xi^{1+2i\alpha}} \qquad for \qquad \xi \to \infty$$

[1]V.E. Zakharov and E A. Kuznetsov, JETP, **64**, 773 (1986).

Self-similar solution for strong collapse[1]

$$\psi = Re^{i\phi}$$

NLS

$$\partial_t R^2 + \nabla \cdot (R^2 \nabla \phi) = 0$$

$$\partial_t \phi + (1/2)(\nabla \phi)^2 = R^2 + (1/2) R^{-1} \nabla^2 R$$

Hydrodynamic approximation

$$\partial_t R^2 + \nabla \cdot (R^2 \nabla \phi) = 0$$

$$\partial_t \phi + (1/2)(\nabla \phi)^2 = R^2$$

- hydrodynamics equation for
gas with the negative pressure and
adiabatic constant = -2

Look for solutions in the following self-similar form

$$|\psi|^2 = \frac{1}{a(t)^3} n(\xi), \qquad \xi = \frac{r}{a(t)}$$

$$n(\xi) = \lambda(1-\xi^2), \qquad \lambda = const$$

$$\phi = \frac{a_t a}{2}\xi^2 + \lambda^2 \int_0^t \frac{dt'}{a(t')^3}$$

$$a(t) = (25/3\lambda^2)^{1/5}(t_0 - t)^{2/5} \qquad for \qquad t \to 0$$

But that solution is unstable!

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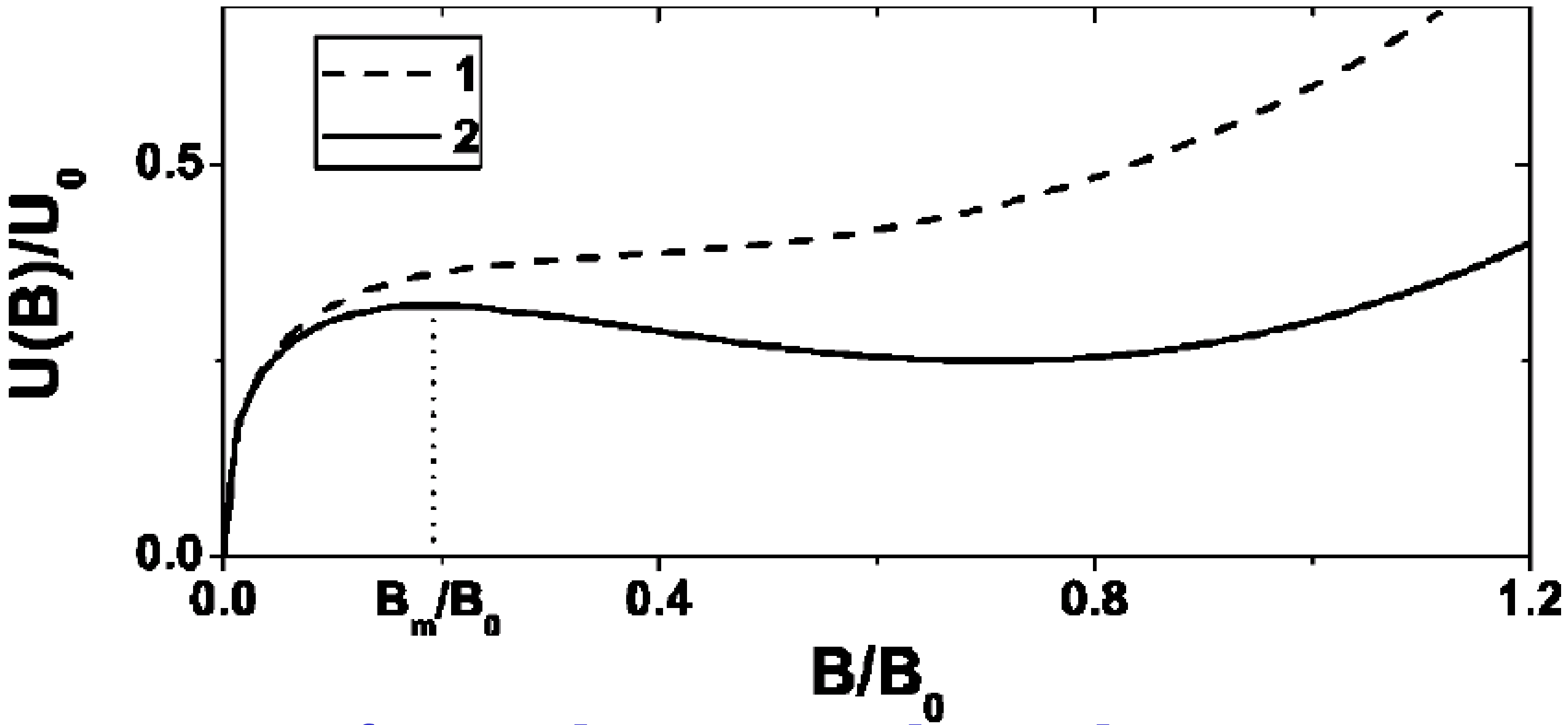
[1]V.E. Zakharov and E A. Kuznetsov, JETP, **64**, 773 (1986).

Basic differential inequality for b=3 and $\omega_0 \neq 0$:

$$\partial_t^2 \langle r^2 \rangle \leqslant \frac{1}{2mN}\left[12E - \frac{\hbar^2}{2m}\left(\frac{9N}{\langle r^2 \rangle} + \frac{m^2 N(\partial_t \langle r^2 \rangle)^2}{\hbar^2 \langle r^2 \rangle} \right) - 10 m \omega_0^2 N F(\gamma) \langle r^2 \rangle \right],$$

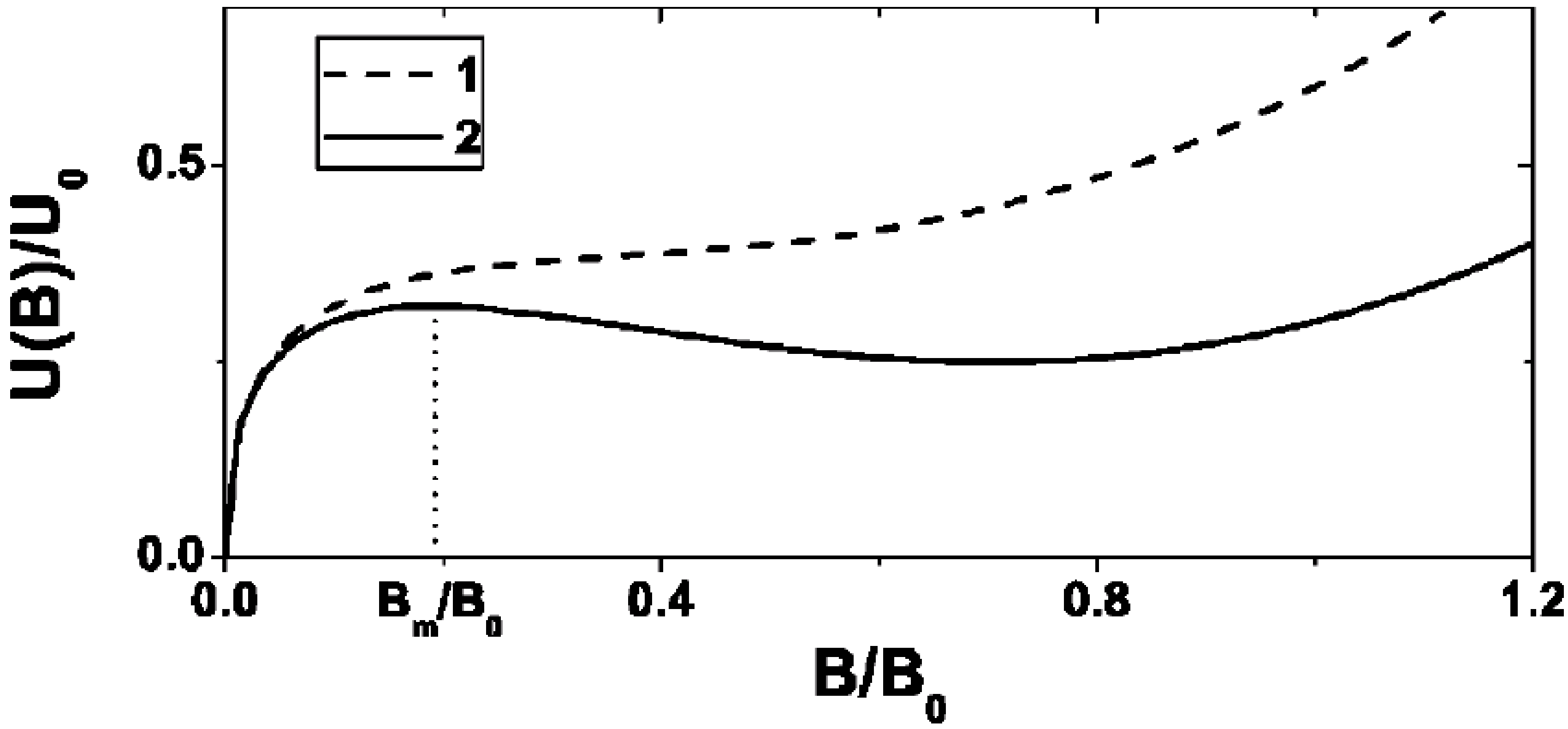
where $F(\gamma) = 1$ for $\gamma \geqslant 1$ and $F(\gamma) = \gamma^2$ for $\gamma < 1$.

Motion of Newtonian "particle" with coordinate ***B*** in potential $U(B)$ with additional nonpotential force $-f^2(t)$:



Energy of particle is time-dependent:

$$\mathcal{E}(t) = B_t^2/2 + U(B)$$



Curve 1: $E \leqslant \hbar\,\omega_0 N[F(\gamma)5]^{1/2}/2 \equiv E_{critical}$

Curve 2: $E > E_{critical}$

Barrier: $B_m^{4/5} = 3(E - [E^2 - E_{critical}^2]^{1/2})/[5m\,\omega_0^2 F(\gamma)]$

Change of variable $\langle r^2\rangle = B^{4/5}/N$ gives:

$$\partial_t^2 B \leqslant \frac{5}{2m}\left[3EB^{1/5} - \frac{\hbar^2}{8m}\frac{9N^2}{B^{3/5}} - \frac{5}{2}m\omega_0^2 F(\gamma)B\right]$$

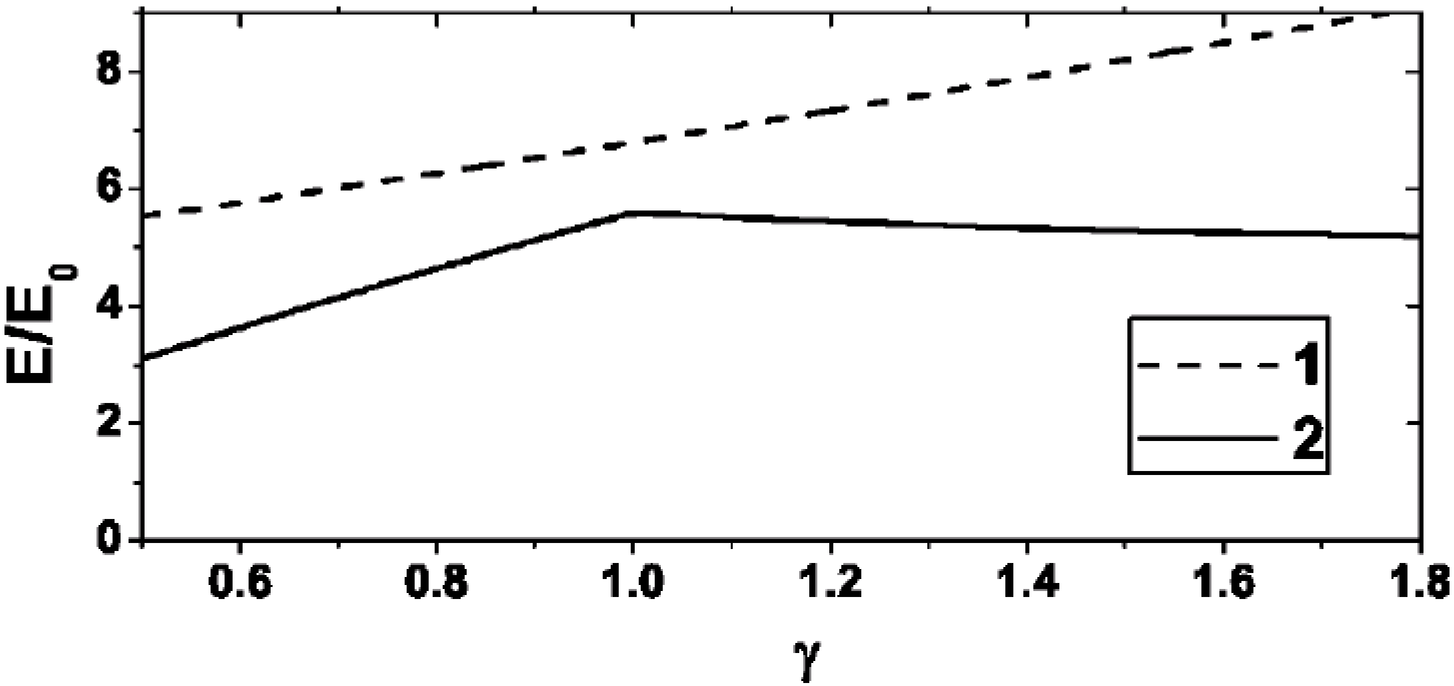
Equivalent form:

$$B_{tt} = -\frac{\partial U(B)}{\partial B} - f^2(t),$$

$$U = -\frac{25}{4m}EB^{6/5} + \frac{\hbar^2 225N^2}{32m^2}B^{2/5} + \frac{25}{8}\omega_0^2 F(\gamma)B^2$$

Comparison with variational approach:

$$\Psi_0 = N^{1/2}\pi^{-3/4}(L_\rho^2 L_3)^{-1/2} e^{-(x_1^2+x_2^2)/2L_\rho^2} e^{-x_3^2/2L_3^2}$$



Curve 1 – variational estimate

Curve 2 – sufficient collapse criterion[1]

$$E_0 = \hbar^{7/2}\omega_0^{1/2}/d^2 m^{3/2}$$

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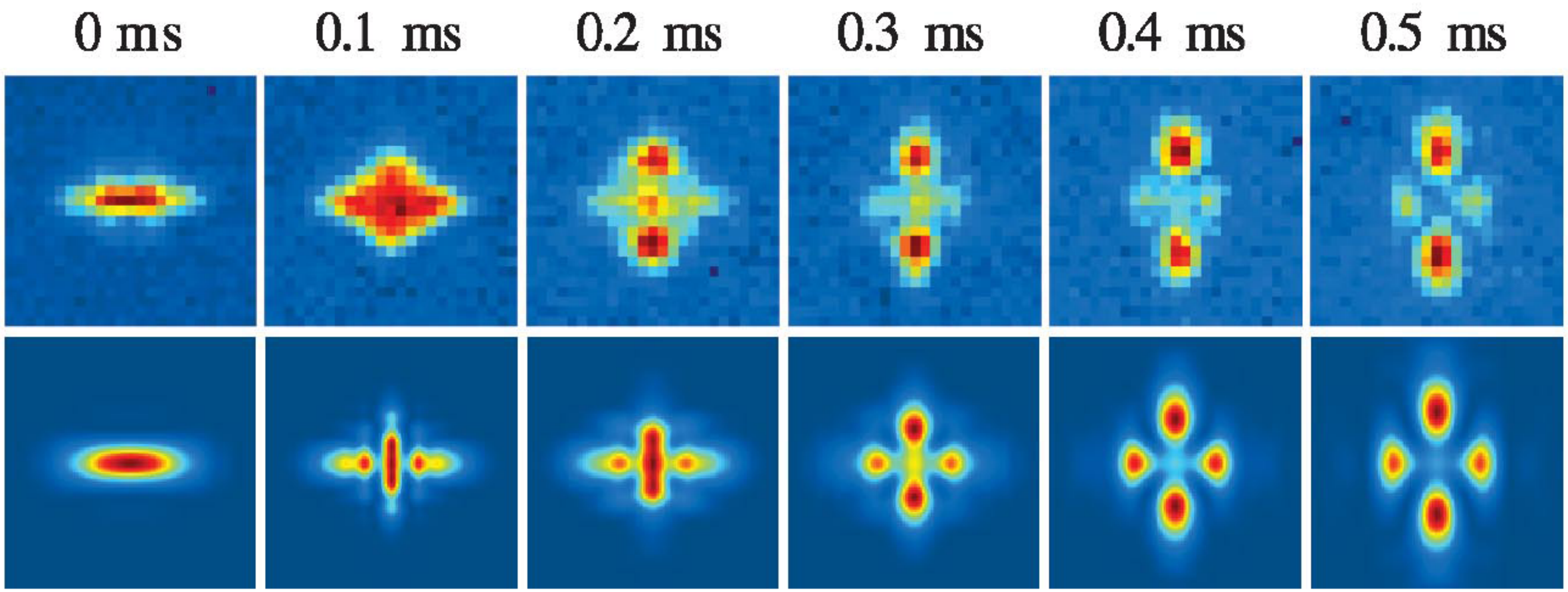
[1] P.M. Lushnikov, Phys. Rev. A. **66**, 051601(R) (2002).

Experimental observation of the dipolar Bose-Einstein condensate collapse



[1] T. Lahae et al. Phys. Rev. Lett. **101**, 080401 (2008).

Strong collapse for dipole-dipole interaction

$$i\frac{\partial\psi(\mathbf{r})}{\partial t} + \nabla^2\psi(\mathbf{r}) - \int \frac{(1-3\cos^2\theta)|\psi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|^3} d\mathbf{r}'\psi(\mathbf{r}) = 0$$

Look for solutions in the following self-similar form[1]

$$|\psi|^2 = \frac{1}{a_\rho(t)^2 a_\zeta(t)} n(\xi,\zeta), \qquad \xi = \frac{\rho}{a_\rho(t)}, \quad \zeta = \frac{z}{a_\zeta(t)}$$

$$\Rightarrow$$

$$n(\xi) = \lambda(1-\xi^2-\zeta^2), \qquad \lambda = const$$

$$\phi = f_\xi(t)\xi^2 + f_\zeta(t)\zeta^2 + \phi_0(t)$$

Unstable?

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[1]Ticknor, Parker, Melatos, Cornish, O'Dell, Martin, Phys. Rev. A, **78**, 013615 (2008).

Long-time condensate existence for b=3

Energy functional: $$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\delta E}{\delta\Psi^*}$$

$$E = E_K + E_P + E_{NL} + E_{DD},$$

where

$$E_K = \int \frac{\hbar^2}{2m}|\nabla\Psi|^2 d^3\mathbf{r},$$

$$E_P = \int \frac{1}{2}m\omega_0^2(x_1^2 + x_2^2 + \gamma^2 x_3^2)|\Psi|^2 d^3\mathbf{r},$$

$$E_{NL} = \frac{g}{2}\int |\Psi|^4 d^3\mathbf{r},$$

$$E_{DD} = \frac{1}{2}\int |\Psi(\mathbf{r})|^2 V(\mathbf{r}-\mathbf{r}')|\Psi(\mathbf{r}')|^2 d^3\mathbf{r}d^3\mathbf{r}'.$$

Long-time condensate existence for b=3

$$E = E_K + E_P + E_{NL} + E_{DD},$$

Dipole-dipole interaction energy:

$$E_{DD} = (1/2)\int |R_{\mathbf{k}}|^2 V_{\mathbf{k}} d^3\mathbf{k}/(2\pi)^3$$

$$V_{\mathbf{k}} = -(4\pi/3)d^2(1-3\cos^2\alpha)$$

α - angle between ***k*** and ***d***.

$$V_{\mathbf{k}} \geqslant -(4\pi/3)d^2$$

$$E_{DD} \geqslant -(2\pi/3)d^2 Y, \qquad Y \equiv \int |\Psi|^4 d^3\mathbf{r}$$

Embedding theorem:

$$Y \leqslant (4/3^{3/2} N_0) N^{1/2} X^{3/2}, \qquad X \equiv \int |\nabla \Psi|^2 d^3\mathbf{r}, \quad Y \equiv \int |\Psi|^4 d^3\mathbf{r}$$

$N_0 = 18.94$ - number of particles for the ground state solution[1] of nonlinear Schrödinger equation:

$$\phi_0 = \lambda R(\lambda \mathbf{r}) e^{i\lambda^2 t}, \qquad -\lambda^2 R + \nabla^2 R + R^3 = 0, \; N_0 \equiv \int R^2 d^3\mathbf{r}$$

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[1]E. A. Kuznetsov, J. J. Rasmussen , K. Rypdal, and S. K. Turitsyn, Physica D **87**, 273 (1995).

Inequality for energy functional:

$$E = E_K + E_P + E_{NL} + E_{DD}$$

$$\geqslant \frac{\hbar^2}{2m} X + \frac{9 m \omega^2}{8X} F(\gamma) N^2 - \frac{2(4\pi d^2 - 3g)}{3^{5/2} N_0} N^{1/2} X^{3/2} \equiv E_l(X).$$

$$X \equiv \int |\nabla \Psi|^2 d^3 \mathbf{r}$$

$4\pi d^2 \leqslant 3g$ - global existence as ***X*** is finite[1]

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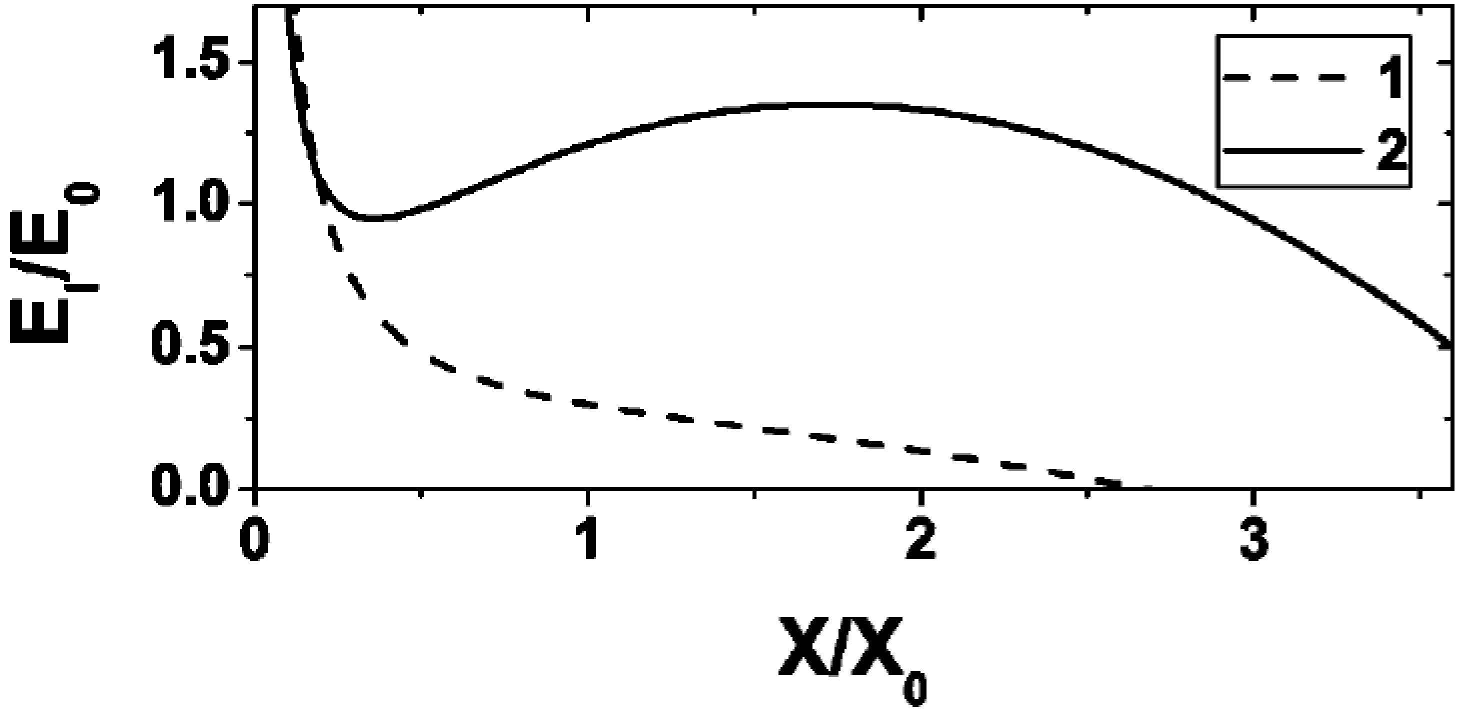
[1] M.I. Weinstein, Commun. Math. Phys. **87**, 567 (1983).

Global existence for $4\pi d^2 > 3g$.

$$E \geqslant \frac{\hbar^2}{2m}X + \frac{9m\omega^2}{8X}F(\gamma)N^2 - \frac{2(4\pi d^2 - 3g)}{3^{5/2}N_0}N^{1/2}X^{3/2} \quad \equiv E_l(X).$$



Curve 1: $N > N_c\,,\ N_c \equiv 2^{3/2}3\hbar^{5/2}N_0/[5^{5/4}(4\pi d^2 \ -3g)F(\gamma)^{1/4}m^{3/2}\omega^{1/2}]$

Curve 2: $N < N_c$ - global existence for ***X*** left from barrier

Generation of gravity-like potential[1]

Dipole-dipole interaction induced by a lser beam of intesity ***I***, Wave vector ***q*** and polarization $\hat{\mathbf{e}}$:

$$U(\mathbf{r}) = \left(\frac{I}{4\pi c \varepsilon_0^2}\right)\alpha^2(q)\hat{\mathbf{e}}_i^*\hat{\mathbf{e}}_j V_{ij}(q,\mathbf{r})\cos(\mathbf{q}\cdot\mathbf{r})$$

$$V_{ij} = \frac{1}{r^3}[(\delta_{ij} - 3\hat{\mathbf{r}}_i\hat{\mathbf{r}}_j)(\cos qr + qr\sin qr) - (\delta_{ij} - \hat{\mathbf{r}}_i\hat{\mathbf{r}}_j)q^2r^2\cos qr],$$

$$\hat{\mathbf{r}}_i = r_i/r$$

$\alpha(q)$ - isotropic dynamic polarizability at frequency cq

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[1]O'Dell, Giovanazzi, Kurizki, and Akulin, Phys. Rev. Lett. **84**, 5687 (2000).

For 3 orthogonally polarized laser beams pointing in $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ directions we expand in small qr

$\Rightarrow$

$$U(\mathbf{r}) = -\frac{3Iq^2\alpha^2}{(16\pi c\varepsilon_0^2)} \times \frac{1}{r}\left[\frac{7}{3} + (\sin\theta\cos\phi)^4 + (\sin\theta\sin\phi)^4 + (\cos\theta)^4\right].$$