Vortex structures in rotating Bose-Einstein condensates

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OUTLINE

- Analytical solution for the vortex lattice in a rapidly rotating trapped 2D Bose-Einstein condensate in the lowest Landau level approximation. Cases of symmetric and asymmetric external harmonic potential are considered.

- Exact spectrum for a low energy excitations of vortex lattice (Tkachenko modes) is found. The dumping due to the quasiparticles decay is calculated.

- The density matrix is calculated. It has an algebraic decay, indicating on the absence of long-range order in the vortex lattice already at zero temperature.

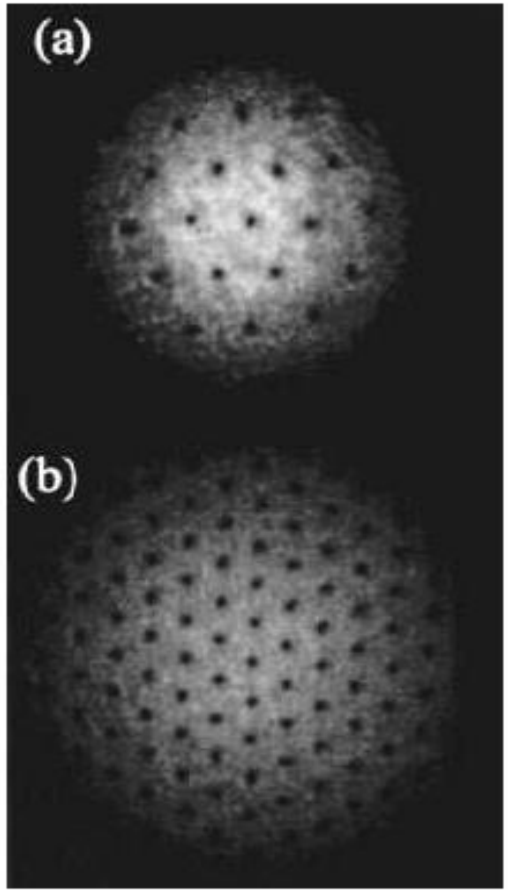


FIG. 12 Images of expanded ^{87}Rb condensates showing (a) small vortex array for slow rotation and (b) large vortex array for rapid rotation. Note the highly regular triangular form. Adapted from Coddington *et al.* (2004).

Gross-Pitaevskii equationin the lowest Landau level. Solution for a symmetric harmonic potential

$$\frac{\hat{\mathbf{p}}^2}{2m}\psi + g|\psi|^2\psi + V(\mathbf{r})\psi - \Omega\hat{L}_z\psi = \mu\psi$$

$$H = \frac{\hat{\mathbf{p}}^2}{2m} - \Omega\hat{L}_z + \frac{1}{2}m\omega^2 r^2 = \frac{1}{2m}(\hat{\mathbf{p}} - e\mathbf{A}/c)^2 + \frac{1}{2}m(\omega^2 - \Omega^2)r^2$$

$$\mathbf{A} = [\mathbf{B}\times\mathbf{r}]/2 = m[\mathbf{\Omega}\times\mathbf{r}]$$

Lowest Landau Level approximation:

$$\Psi = f(z)e^{-\omega z\bar{z}/2}$$

$$Hf(z) = (\omega - \Omega)z\partial_z f(z)$$

•LLL projection: $\hat{P} = \sum |n\rangle\langle n|$ $\Psi_n(z, \bar{z}) = \frac{z^n}{l\sqrt{\pi n!}} \exp\left[-\frac{z\bar{z}}{2}\right]$

$$\widehat{P} f(z, \overline{z}) = \frac{1}{\pi} \int dz' d\overline{z}' \exp[-z'\overline{z}' + z\overline{z}'] f(z', \overline{z}') =$$
$$: f(z, \frac{\partial}{\partial z}) :$$

$$\hbar(\omega - \Omega) z \partial_z f(z) + \frac{Ng}{\pi l^2} \int dz' d\bar{z}' |f(z')|^2 f(z') \exp(z\bar{z}' - 2z'\bar{z}') = \tilde{\mu} f(z)$$

$$\widehat{P}[((\omega - \Omega)|z|^2 - \mu + \frac{Ng}{\pi l^2} |\Psi|^2) f(z)] = 0$$

$\Omega = \omega :\longrightarrow$

•Exact solution:

$$f_0(z) = \frac{(2v)^{1/4}}{\sqrt{S}} \vartheta_1\left(\pi z/b_1, \tau\right) \exp(\pi z^2/2v_c)$$

$$\vartheta_1(\zeta, \tau) = \frac{1}{i} \sum_{n=-\infty}^{\infty} (-1)^n \exp\{i\pi\tau(n+1/2)^2 + 2i\zeta(n+1/2)\}$$

$$\tau = u + iv; \qquad v_c = b_1^2 v = \pi$$

zeros of Theta - function: $z_{n,m} = b_1 n + b_1 \tau m$

$$|\vartheta_1(\pi z/b_1, \tau)| \sim e^{\pi y^2/v_c}$$

•Triangle lattice: $u = -1/2, \quad v = \sqrt{3}/2$

$\Omega < \omega :\longrightarrow$

$$f(z) = (2v)^{1/4} \sum_{n=-\infty}^{\infty} (-1)^n \hat{g}(a) \tilde{q}^{a^2} \exp\left[\frac{i\pi}{b_1} az + \frac{z^2}{2}\right]$$

$a = 2n + 1; \quad q = \exp[i\pi\tau]$

$$\left\{(\mu^* - \hat{A}^+\hat{A})\hat{g}(a) - \frac{3^{1/4}\beta}{\sqrt{2}} \sum_{b,c} \hat{g}(a-b)\hat{g}(a-c)\overline{\hat{g}(a-b-c)} \exp\left[-\frac{\pi^2}{4b_1^2}(b^2+c^2)\right](-1)^{mp}\right\}$$

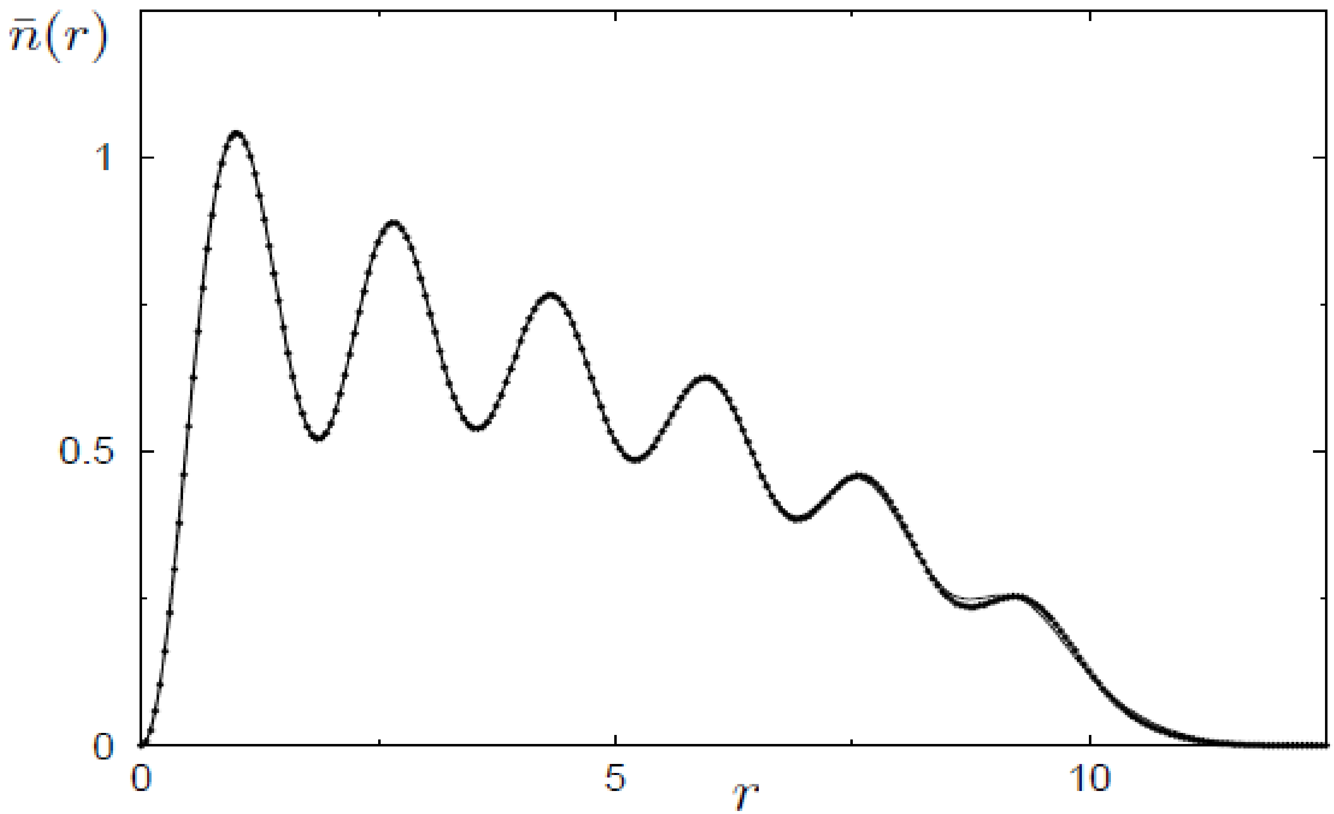
$$\times \tilde{q}^{a^2} \exp\left[\frac{i\pi}{b_1} az\right] = 0. \tag{20}$$

$$\hat{A}, \hat{A}^+ = \frac{\pi}{2b_1} a \pm \frac{b_1}{\pi} \frac{\partial}{\partial a}. \qquad \hat{g}(a) = \frac{1}{\sqrt{\alpha\beta}} \sqrt{R^2 - \hat{A}^+\hat{A}}\, \Theta(R^2 - \hat{A}^\dagger \hat{A}).$$

$$f(z) = \frac{(2v)^{1/4}}{\sqrt{\alpha\beta}} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{[R^2]} (-1)^{[n(n-1)/2]} \sqrt{R^2 - k}\, \frac{(iz)^k}{2^{k/2} k!} H_k\left(\sqrt{\frac{\pi v}{2}}(2n+1)\right) \exp\{-\pi v (2n+1)^2/4\}$$

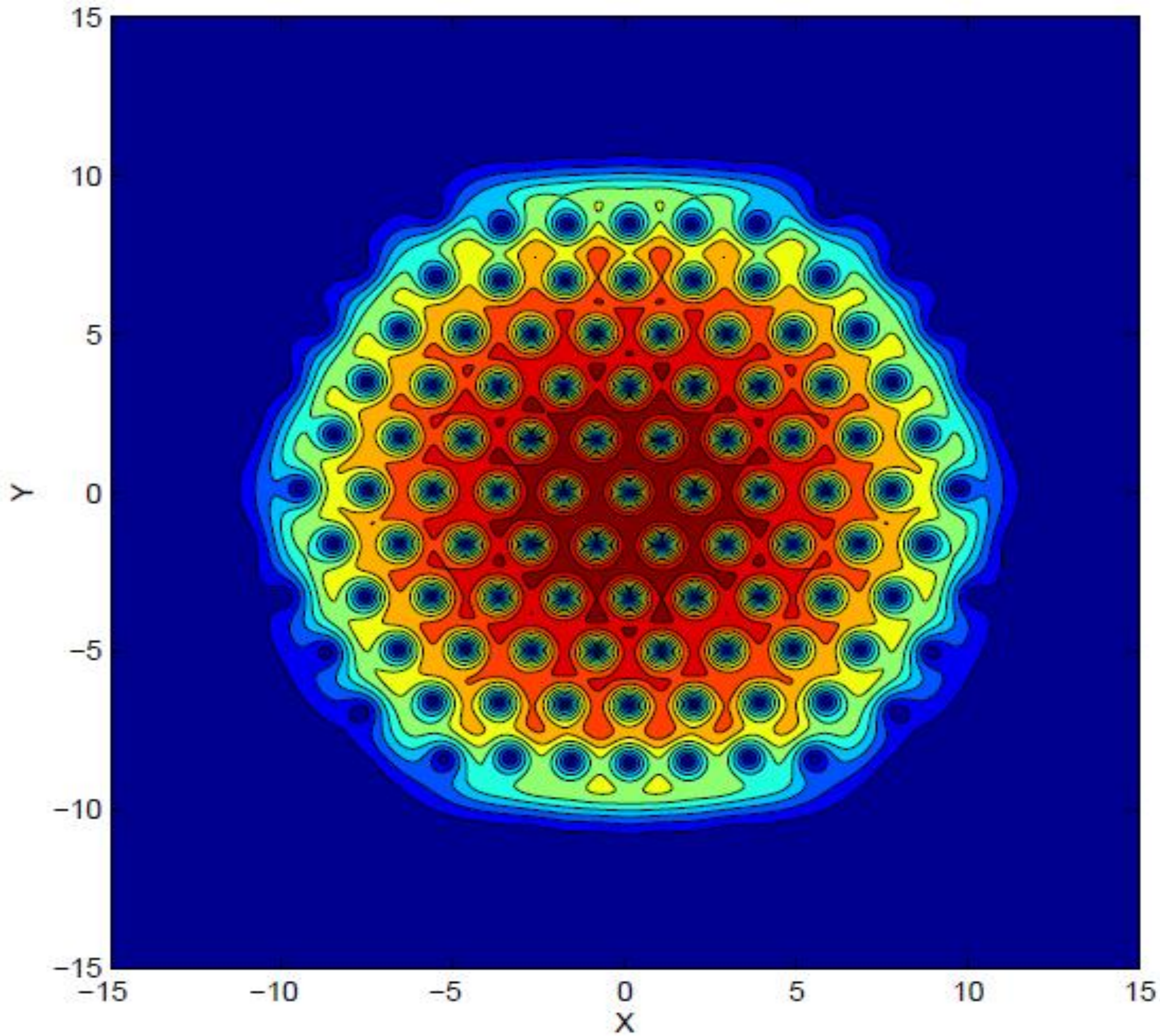
Comparison with numeric simulation

$$\bar{n}(r) = N \int |\psi(z, \bar{z})|^2 \frac{d\varphi}{2\pi}$$



Angular-averaged density $\bar{n}(r)$ in units of n_{2D} versus r (in units of l) for $R = 11$

- **(Solid curve-analytic, filled circles-numeric)**



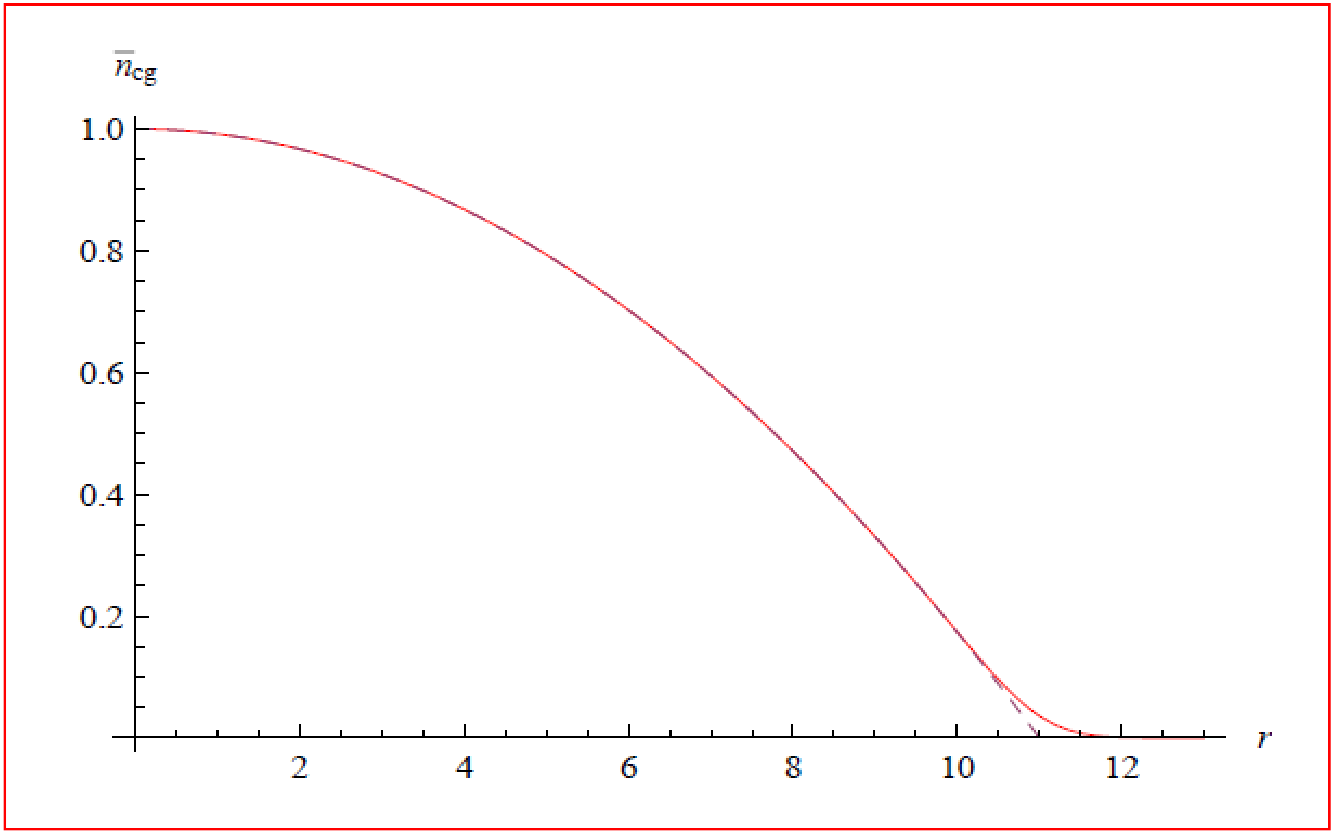
Condensate wave-function $|\psi(x,y)|^2$ for $R = 11$.

•The course grained density:

$$\bar{n}_{cg}(r) = \bar{n}_{2D}\sum_{k=0}^{[R^2]}\left(1-\frac{k}{R^2}\right)\frac{r^{2k}}{k!}\exp(-r^2) = \bar{n}_{2D}\left(1-\frac{r^2}{R^2}\right)\frac{\Gamma(R^2,r^2)}{\Gamma(R^2)} + \frac{r^{2R^2}\exp(-r^2)}{R^2\Gamma(R^2)}$$

$$\bar{n}_{cg}(r) = \bar{n}_{2D}\left(1-\frac{r^2}{R^2}\right)\Theta(R-r)$$

$$\bar{n}_{cg}(r) = \frac{\bar{n}_{2D}}{\sqrt{2\pi R^2}}\frac{\exp[-2(r-R)^2]}{4(r-R)^2}; \text{ at } R^{1/3} \gg (r-R) \gg 1$$



•Fig. The coarse grained density

1D geometry:

$$V(\mathbf{r}) = m(\omega_x^2 x^2 + \omega_y^2 y^2)/2, \text{ with } \omega_y < \omega_x.$$

$$\boxed{\Omega = \omega_y}$$

- LLL approximation:

$$\Psi = [f(\zeta)/\tilde{l}]\exp\left(-\tilde{y}^2\right).$$

$$\tilde{f}(\zeta) = \frac{1}{\pi}\int d\zeta' d\bar{\zeta}' f(\zeta', \bar{\zeta}')\exp\{\zeta\bar{\zeta}' + \zeta'^2/2 - \zeta'\bar{\zeta}' - \zeta^2/2\}$$

$$\tilde{x} = -\frac{\tilde{\Omega} y}{\Omega \tilde{l}}; \quad \tilde{y} = \frac{x}{\tilde{l}}; \quad \zeta = \tilde{x} + i\tilde{y}$$

$$-\hbar\omega_0 f''(\zeta) + \frac{Ng}{\pi\tilde{l}^2}\int d\zeta' d\bar{\zeta}' |f(\zeta')|^2 f(\zeta') \exp\left(-2\zeta'\bar{\zeta}' + \zeta\bar{\zeta}' + \zeta'^2 + \frac{\bar{\zeta}'^2}{2} - \frac{\zeta^2}{2}\right) = \tilde{\mu} f(\zeta)$$

- **The solution (ansatz):**

$$f(\zeta) = \frac{(2v)^{1/4}}{\sqrt{L}} \sum_{n=-\infty}^{\infty} (-1)^n g(2n+1) \exp\left(i\pi\tau\frac{(2n+1)^2}{4} + \frac{i\pi\zeta(2n+1)}{b_1}\right)$$

$$\Downarrow$$

$$g(a)\left(\mu^* - \frac{\pi^2 a^2}{b_1^2}\right) = 3^{1/4}\sqrt{2\pi}\tilde{\beta}\sum_{b,c} g(a-b)g(a-c)\overline{g(a-b-c)} \exp\left[-\frac{\pi^2}{4b_1^2}(b^2+c^2)\right](-1)^{m_I}$$

$$\Downarrow$$

$$g(a) = \left(\frac{1}{2\alpha\sqrt{\pi}\tilde{\beta}}\right)^{1/2} \frac{\pi}{b_1}\sqrt{4\tilde{R}^2/\pi v - a^2}\;\Theta\left[\frac{2\tilde{R}}{\sqrt{\pi v}} - a\right]$$

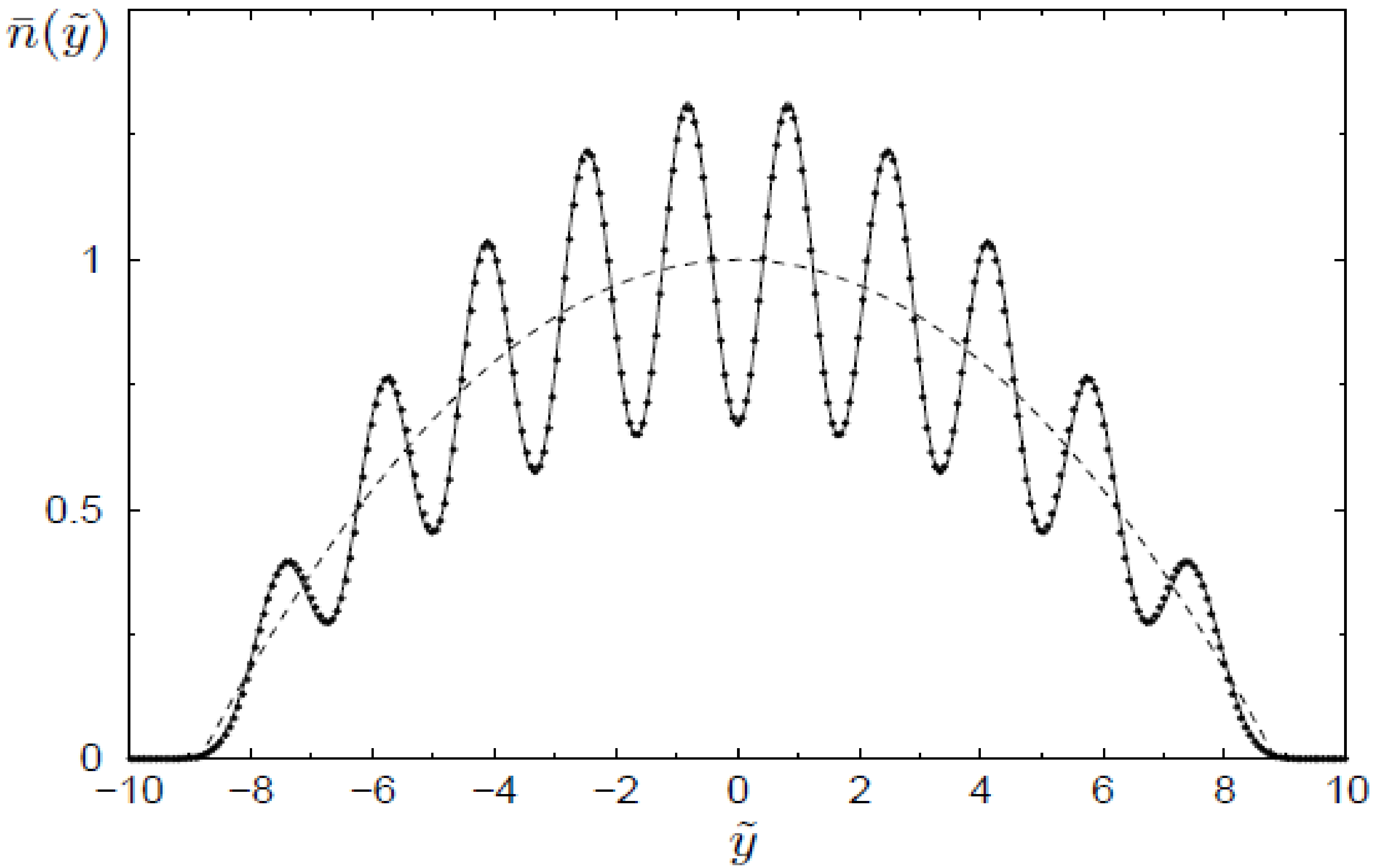
$$\mu^* = 4\tilde{R}^2$$

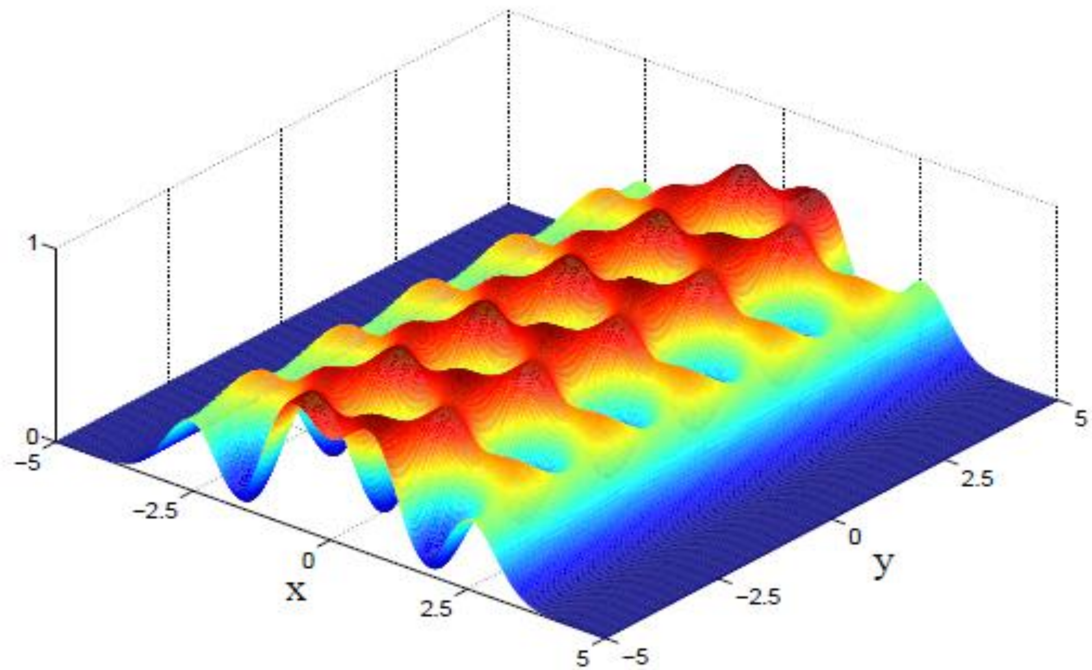
$$\tilde{R} \approx (3\alpha\sqrt{\pi}\tilde{\beta}/8)^{1/3}.$$

•**Line-averaged density analytic(solid curve) and numeric (filled circles) results:**

$$\beta = 900$$

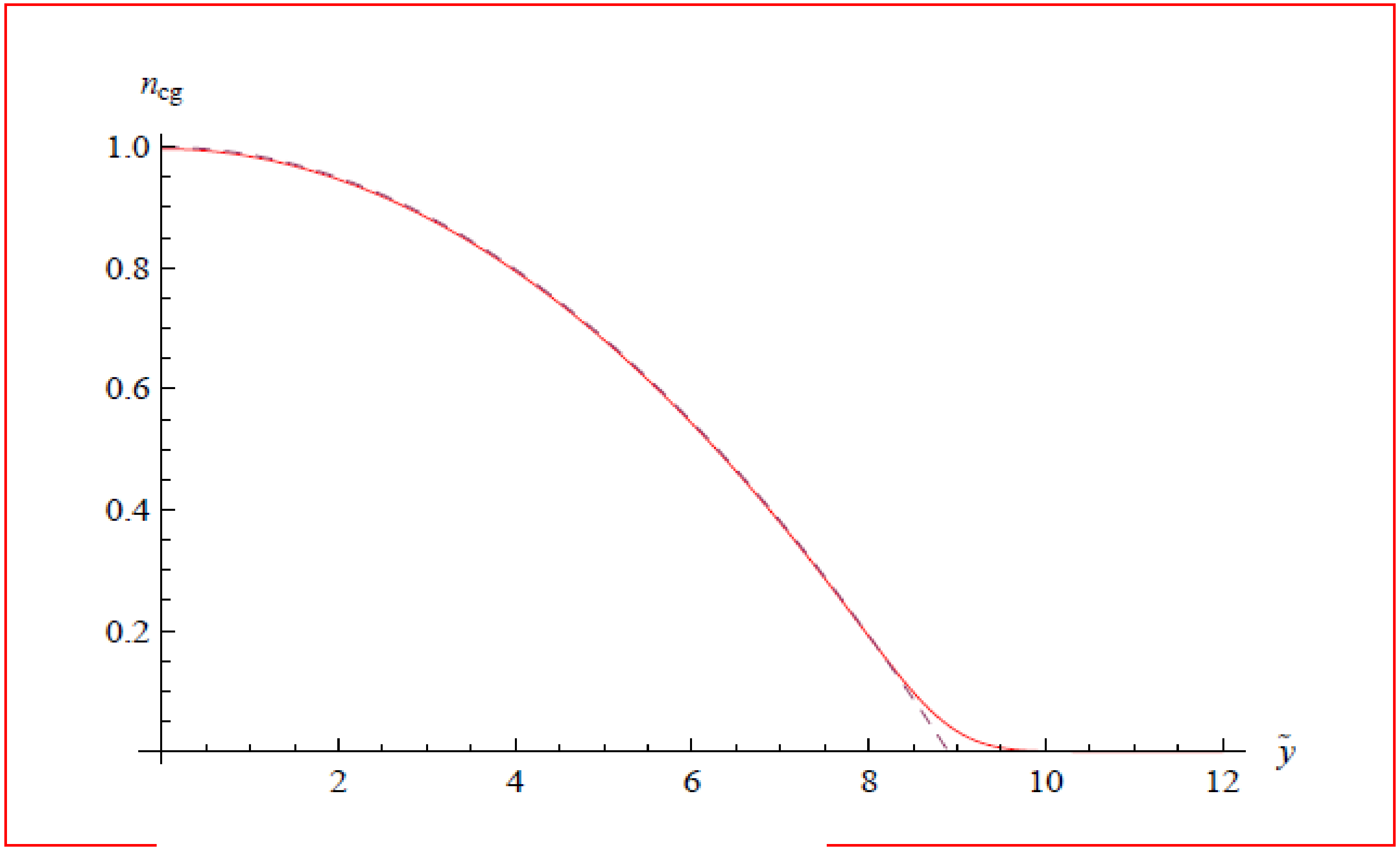
$$\bar{n}(\tilde{y}) = \frac{N}{L}\int |\psi(\tilde{x},\tilde{y})|^2 d\tilde{x}$$





Density profile $|\psi(x,y)|^2$ for $\tilde{\beta} \simeq 50$

•Coarse grained density:



$$n_{cg}(\tilde{y}) = n_{2D}\left(1 - \frac{\tilde{y}^2}{\tilde{R}^2}\right) \qquad \text{for } |y| < R$$

$$n_{cg}(\tilde{y}) = n_{2D}\sqrt{\frac{1}{2\pi\tilde{R}^2}}\,\frac{1}{4(|\tilde{y}| - \tilde{R})^2}\exp\{-2(|\tilde{y}| - \tilde{R})^2\} \qquad \text{for } y - R \gg 1$$

Phase diagram for a condensate in the narrow channel (numerical results)

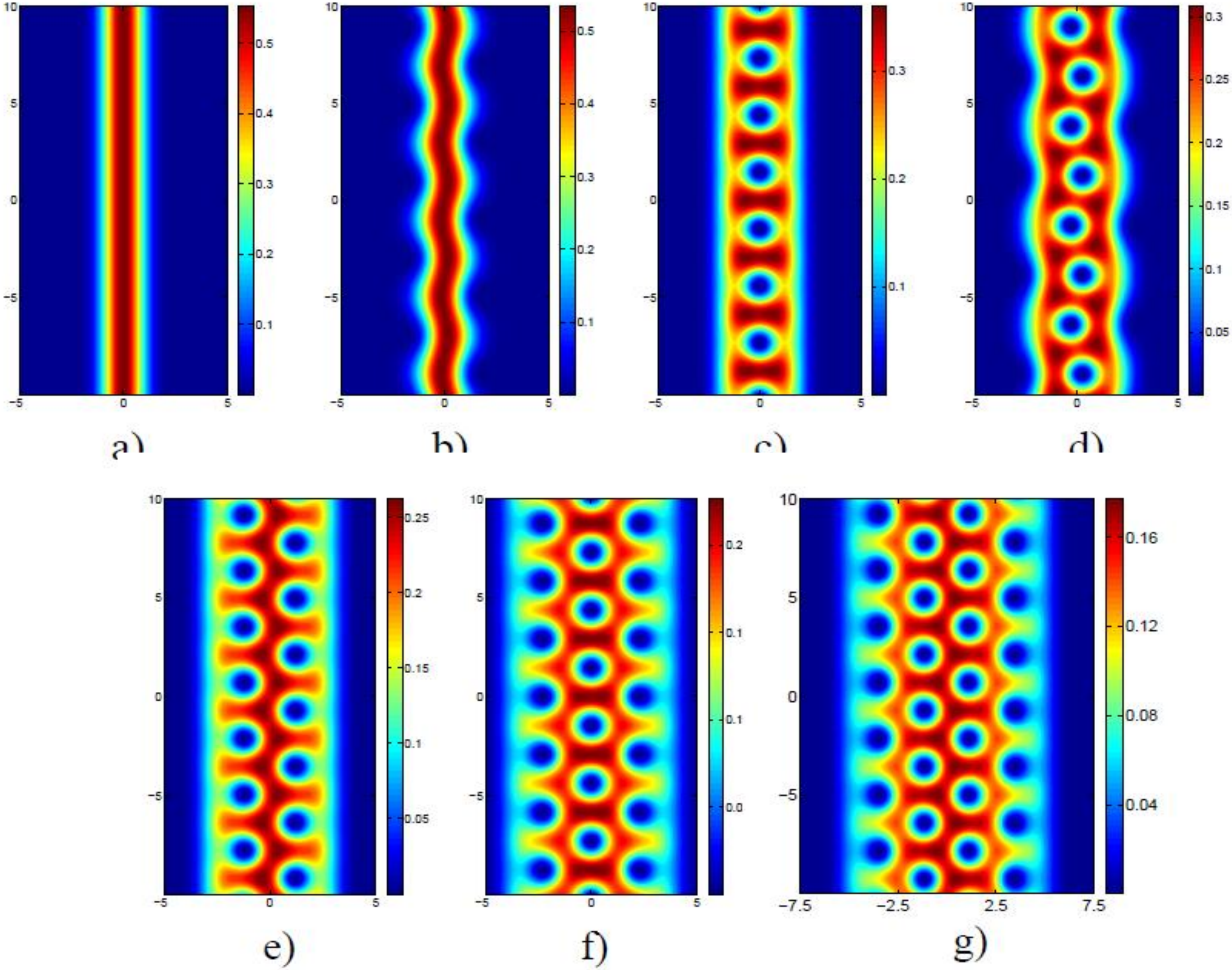
- Numerical minimization of the energy :

$$E(\beta) =$$

$$\sum_k k^2 |a_k|^2 + \tilde{\beta} \sum_{k,k',q} a^*_{k+q} a^*_{k'-q} a_{k'} a_k \exp\left[-\frac{1}{4} \{(k-k'+q)^2 + q^2\} \right]$$

$$\psi(x,y) = \sum_k a_k \Psi_k(\tilde{x}, \tilde{y}).$$

Fig. Condensate wave-function $|\psi(x,y)|^2$ for different values of the interaction strength: a) $\beta = 0$, b) $\beta = 5.2$, c) $\beta = 10$, d) $\beta = 19.2$, e) $\beta = 30$, f) $\beta = 50$, g) $\beta = 100$.



a)
b)
c)
d)
e)
f)
g)

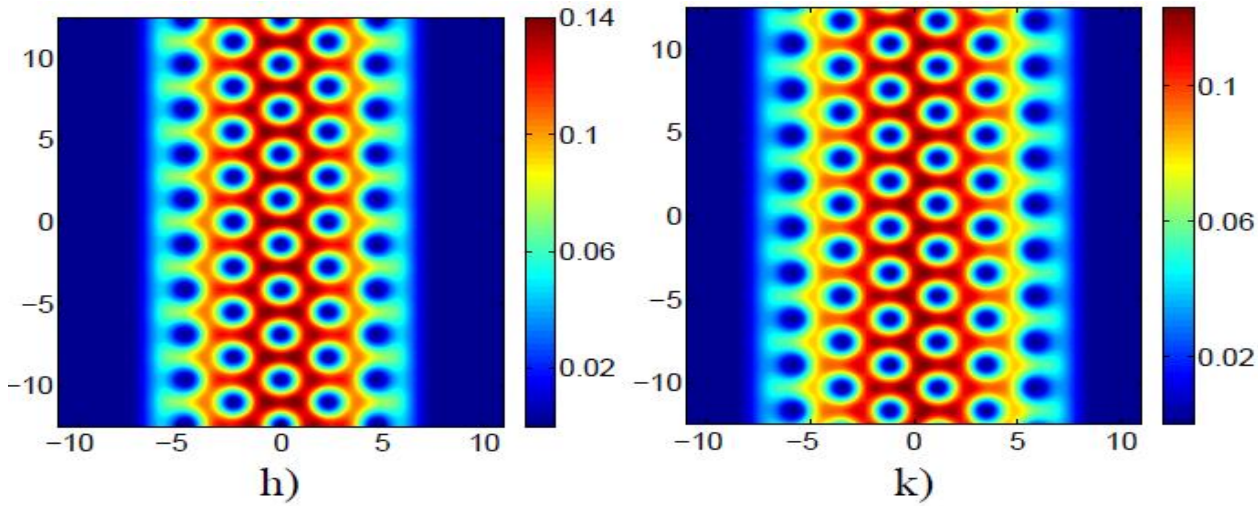
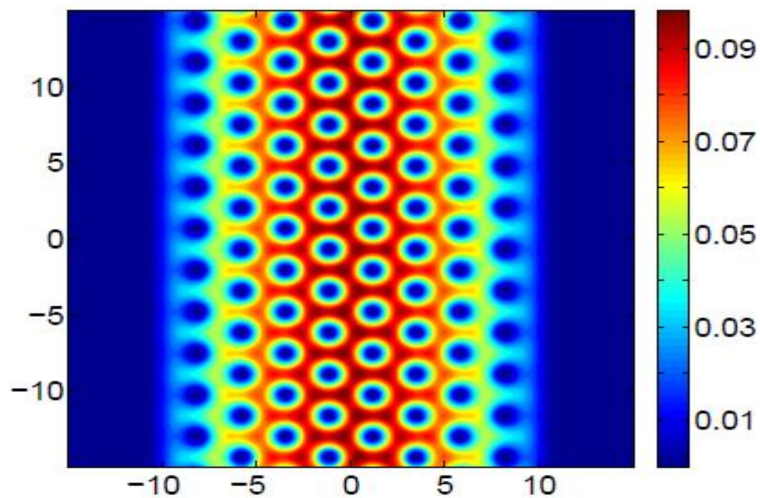


FIG. 9: The same as in Fig. 7 for: h) $\beta = 200$, k) $\beta = 300$.



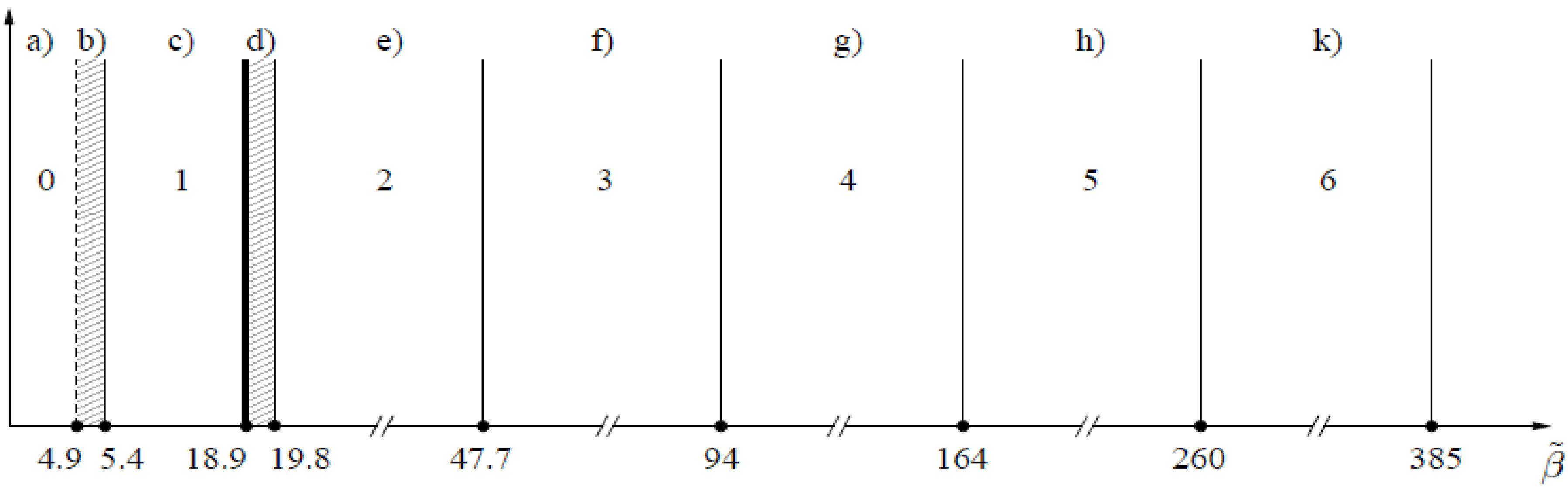


FIG. 11: Zero temperature phase diagram for a rapidly rotating condensate in the narrow channel. Solid vertical lines indicate the points of first order transitions, and the dashed line the point of the second order transition. The bold solid line shows the transition between the states c) and d) (see text). The numbers from 0 to 6 stand for the number of vortex rows in a given range of $\tilde{\beta}$, and the filled areas correspond to corrugated/density-wave states. The letters from a) to k) indicate the figure in which a given vortex state is shown.

$\tilde{R} \approx (3\alpha\sqrt{\pi}\tilde{\beta}/8)^{1/3}$.

$\beta_n/\beta_{n-1} \simeq [(n+1)/n]^3$

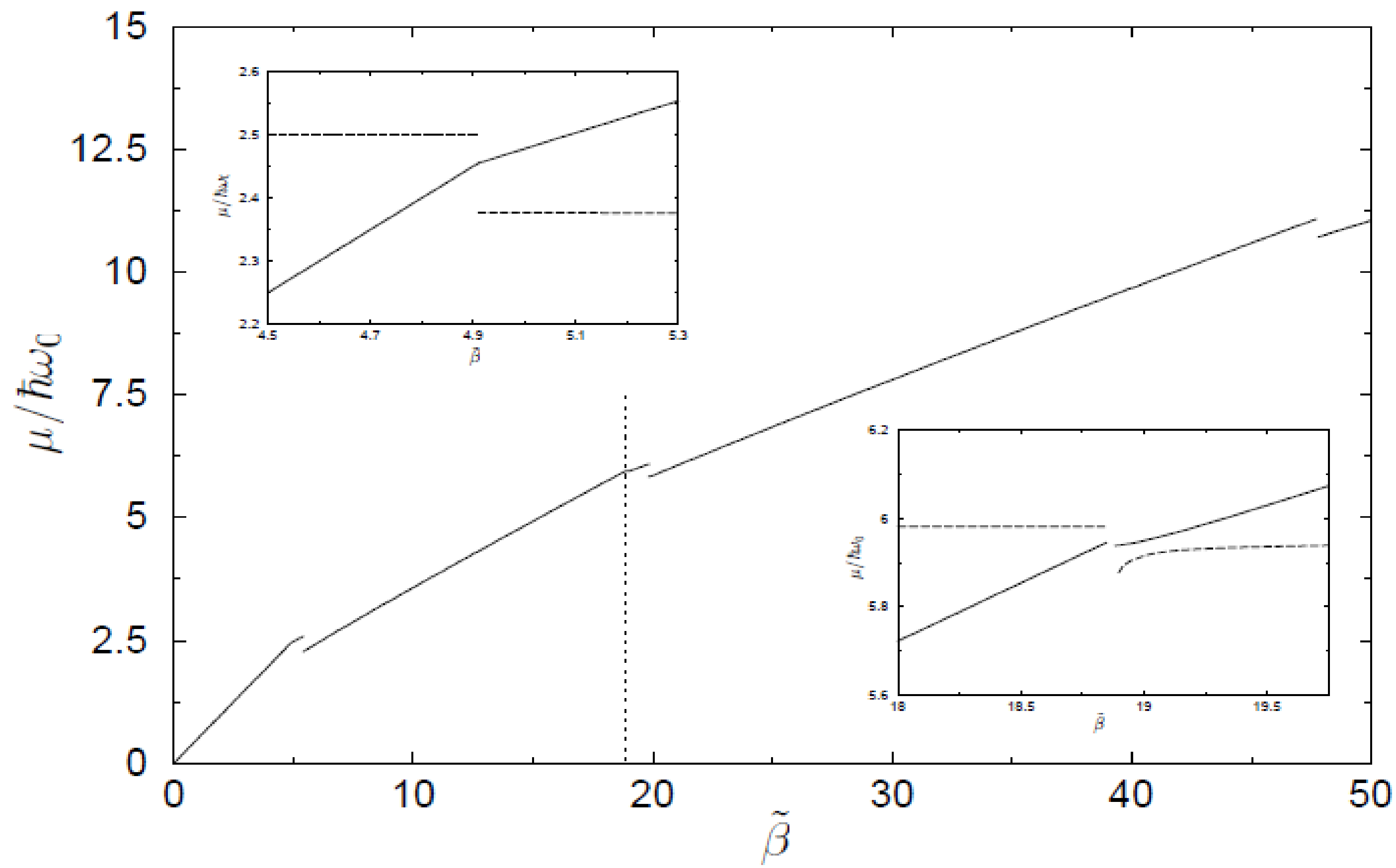


FIG. 12: Chemical potential in units of $\hbar\omega_0$ as a function of β. The dotted line indicates the transition between the states c) and d) (see text). The insets show the dependence $\tilde{\mu}(\tilde{\beta})$ in the vicinity of the quantum transitions at $\tilde{\beta} = 4.9$ (upper inset) and at $\tilde{\beta} \simeq 18.9$ (lower inset). The dashed lines in the insets indicate the derivative $\partial\tilde{\mu}/\partial\tilde{\beta}$ in arbitrary units.

Solution for asymmetric harmonic potential

$$V(x,y) = (\omega_x x^2 + \omega_y y^2)/2, \qquad \Omega < \omega_x, \omega_y$$

$$(\mu - \omega_t^+) f(\zeta) = \frac{\omega_t^+ - \omega_t^-}{2}\left(-\sinh(2\nu) f''(\zeta)\right.$$
$$\left.+2\zeta f'(\zeta)\right) + g\frac{1}{\pi}\mathrm{e}^{-\tanh(\nu)\zeta^2/2}$$
$$\times \int d\zeta' d\bar{\zeta}' \mathrm{e}^{-2\zeta'\bar{\zeta}' + \zeta\bar{\zeta}' + \tanh(\nu)\zeta'^2 + \tanh(\nu)\bar{\zeta}'^2/2} f^2(\zeta')\overline{f(\zeta')}.$$

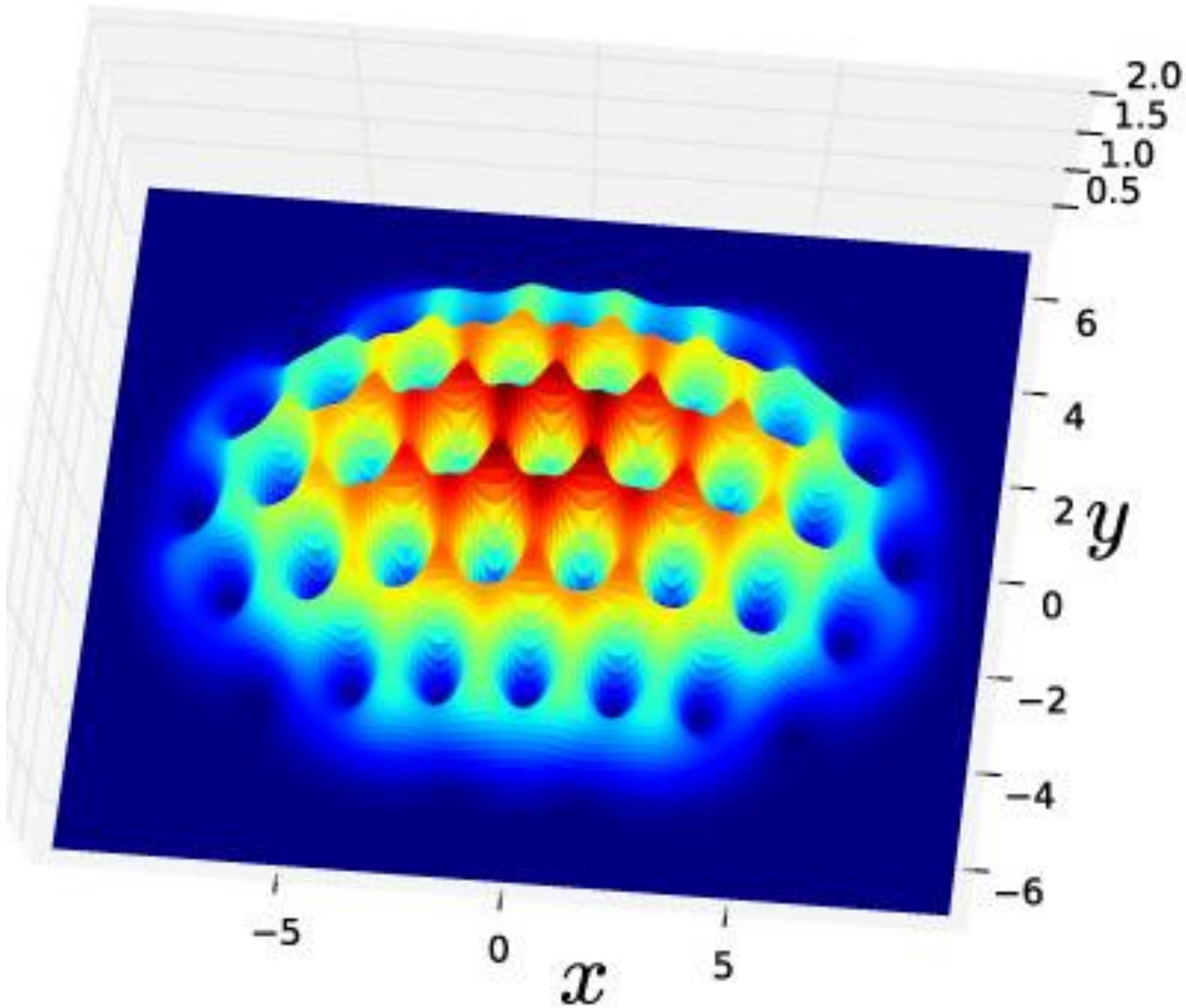
$$\zeta = \tilde{x} + i\tilde{y} = \frac{i}{2}\sqrt{\omega_t^+ - \omega_t^-}\sqrt{\sinh 2\nu}\left[z\rho - \frac{\bar{z}}{\rho}\right], \qquad z = x + iy,$$

$$\tanh\nu = \frac{\omega_t^+}{\omega_t^-}\sqrt{\frac{\omega_t^{-2} - \omega_c^2}{\omega_t^{+2} - \omega_c^2}}, \qquad \rho^2 = \sqrt{\frac{(\omega_t^- + \omega_c)(\omega_t^+ + \omega_c)}{(\omega_t^- - \omega_c)(\omega_t^+ - \omega_c)}},$$

$$\tilde{\omega}_x^2 = \omega_x^2 - \omega_c^2,\ \tilde{\omega}_y^2 = \omega_y^2 - \omega_c^2,\ \omega_t^\pm = \sqrt{\omega_c^2 + \left(\frac{\tilde{\omega}_x \pm \tilde{\omega}_y}{2}\right)^2},$$

•The solution:

$$f(\zeta) = \frac{(2v)^{1/4}\sqrt{(1+\tanh\nu)}}{\sqrt{\alpha\beta}} \sum_{n=-\infty}^{\infty} \sum_{k=0}^{[R^2]} (-1)^{[n(n-1)/2]}$$
$$\times \sqrt{R^2-k}\,\frac{(i\sqrt{\tanh\nu})^k}{2^k k!}$$
$$\times H_k\left(\frac{\zeta}{\sqrt{\sinh 2\nu}}\right) H_k\left(\sqrt{\frac{\pi\tilde{v}}{2}}a\right) \mathrm{e}^{-\pi\tilde{v}a^2/4},$$

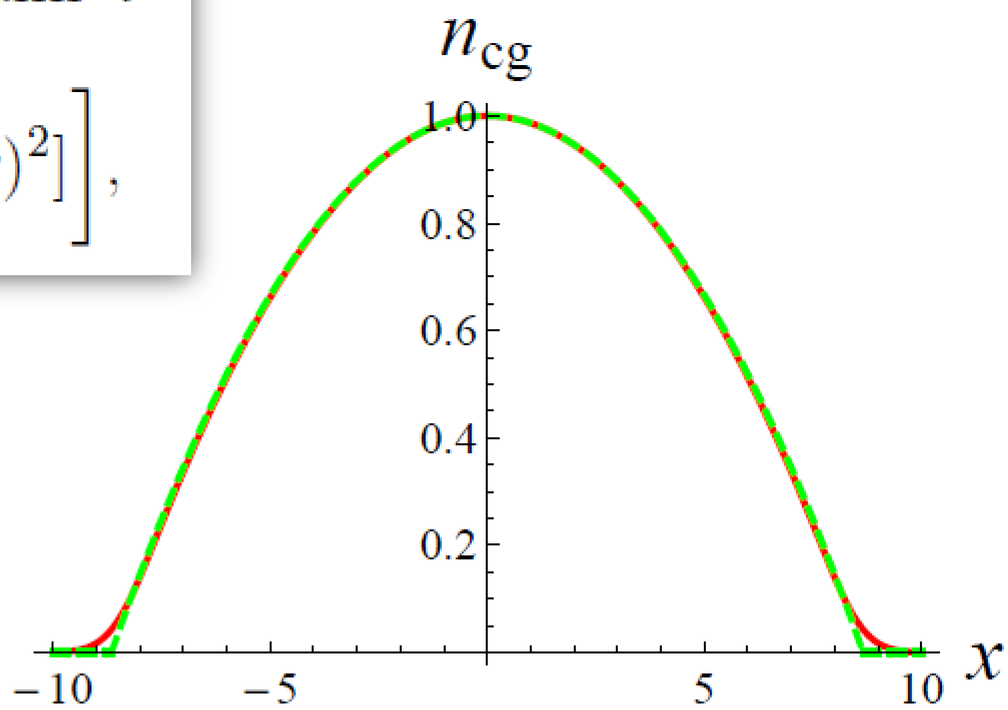


Condensate wave-function $|\psi(x,y)|^2$ for $R = 7$, $\tanh \nu = 1/4$. Coordinates x and y are given in units of l.

•Density profile:

$$\bar{n}_{cg}(r) = \frac{n_{cg}(0)}{\cosh\nu} \sum_{k=0}^{[R^2]} \left(1 - \frac{k}{R^2}\right) \frac{(\tanh\nu)^k}{2^k k!} H_k\left(\frac{\zeta}{\sqrt{\sinh 2\nu}}\right)$$
$$\times H_k\left(\frac{\bar{\zeta}}{\sqrt{\sinh 2\nu}}\right) \mathrm{e}^{-|\zeta|^2 + (\zeta^2 + \bar{\zeta}^2)\tanh(\nu)/2}$$

$$n = n_{cg}(0)\Big[1 - \frac{1}{R^2(1-\tanh^2\nu)}[\tanh^2\nu$$
$$+ \tilde{x}^2(1-\tanh\nu)^2 + \tilde{y}^2(1+\tanh\nu)^2]\Big],$$



Exact spectrum for a low energy excitations of vortex lattice (Tkachenko modes)

$$\Psi(z) = \mathrm{e}^{-|z|^2/2}\left(\sqrt{N} f_0 + \sum_k [u_k(z)\hat{a}_k \mathrm{e}^{-i\epsilon_k t} - \tilde{v}(z)\hat{a}_k^\dagger \mathrm{e}^{i\epsilon_k t}]\right)\mathrm{e}^{-i\mu t}$$

- **Bogoliubov equations**:

$$\begin{aligned} 2\beta P|\Psi_0|^2 u - \beta\Psi_0^2\tilde{v}^* &= (\mu+\epsilon)u \\ 2\beta P|\Psi_0|^2 \tilde{v} - \beta\Psi_0^2 u^* &= (\mu-\epsilon)\tilde{v} \end{aligned}$$

- **Solution:**

$$u_k \mathrm{e}^{-|z|^2/2} = c_1 \Psi_0(z + ik_+/2)\mathrm{e}^{-k^2/8}\mathrm{e}^{i\mathbf{kr}/2}$$

$$\tilde{v}_k \mathrm{e}^{-|z|^2/2} = c_2 \Psi_0(z - ik_+/2)\mathrm{e}^{-k^2/8}\mathrm{e}^{-i\mathbf{kr}/2}$$

•**The spectrum:**

$$\epsilon^2(\mathbf{k})/\beta^2 = [2K_1(\mathbf{k}) - K_0]^2 - |K_2(\mathbf{k})|^2$$

$$K_1(\mathbf{k}) = \frac{e^{-\frac{k_y{}^2}{4}}\left(\vartheta_3\left(\frac{k_x\sqrt{\pi}}{2\sqrt{v}}, e^{-\frac{\pi}{v}}\right)\vartheta_3\left(-\frac{ik_y\sqrt{\pi}}{4\sqrt{v}}, e^{-\frac{\pi}{4v}}\right) + \vartheta_2\left(\frac{k_x\sqrt{\pi}}{2\sqrt{v}}, e^{-\frac{\pi}{v}}\right)\vartheta_4\left(-\frac{ik_y\sqrt{\pi}}{4\sqrt{v}}, e^{-\frac{\pi}{4v}}\right)\right)}{2v}$$

$$K_2(\mathbf{k}) = \frac{1}{2v}e^{-\frac{1}{2}k_y(k_y+ik_x)}\left[\vartheta_3\left(\frac{(k_x-ik_y)\sqrt{\pi}}{2\sqrt{v}}, e^{-\frac{\pi}{v}}\right)\vartheta_3\left(\frac{(k_x-ik_y)\sqrt{\pi}}{4\sqrt{v}}, e^{-\frac{\pi}{4v}}\right) + \vartheta_2\left(\frac{(k_x-ik_y)\sqrt{\pi}}{2\sqrt{v}}, e^{-\frac{\pi}{v}}\right)\vartheta_4\left(\frac{(k_x-ik_y)\sqrt{\pi}}{4\sqrt{v}}, e^{-\frac{\pi}{4v}}\right)\right]$$

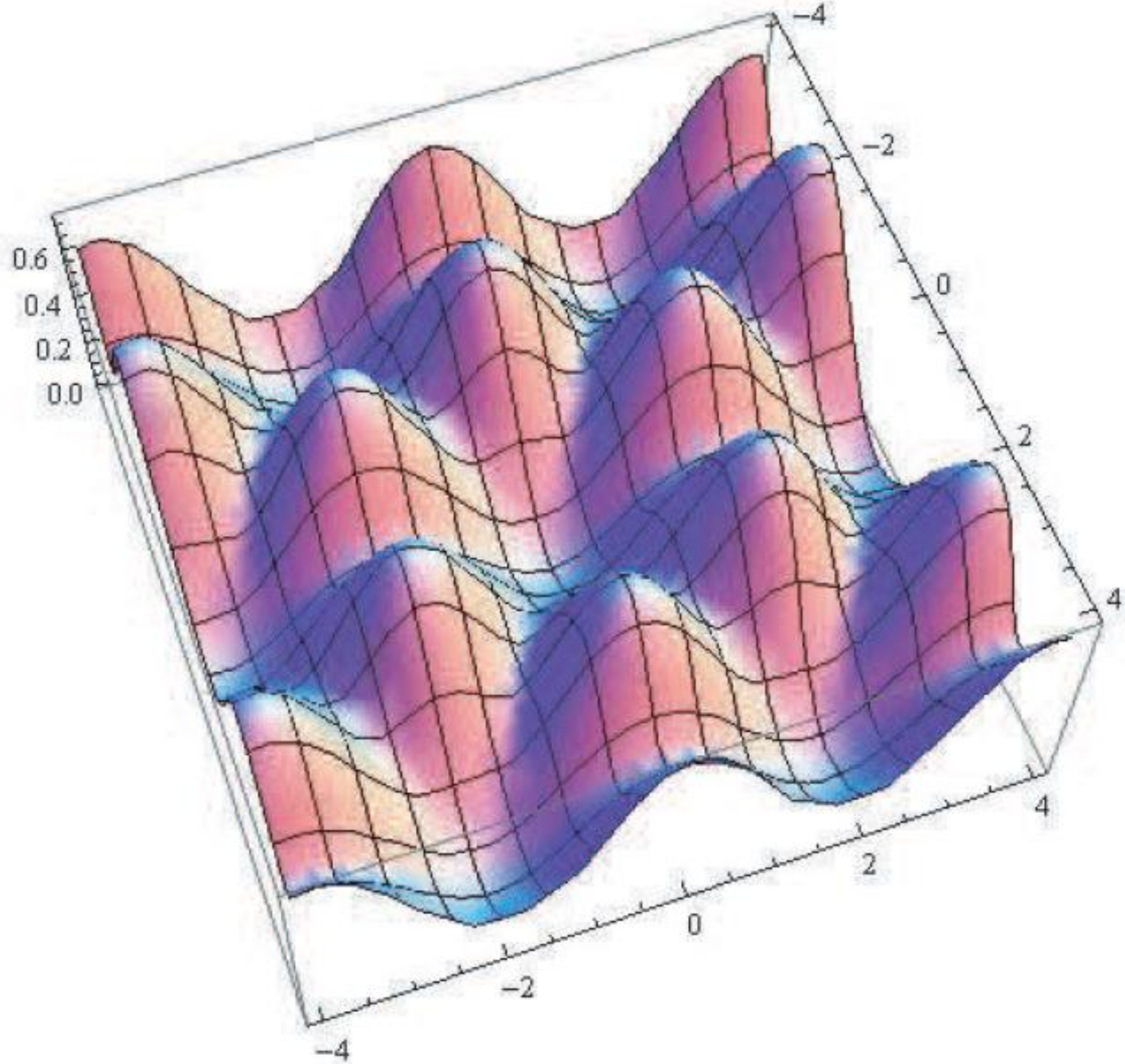
•**For small k**:

$$\epsilon = C_\epsilon \beta k^2 = 0.2824\beta k^2$$

•**Damping of Tkachenko mode:**

$$\hat{V} = \frac{\beta}{\sqrt{N}}\int dx\,dy[f_0^* u_Q^* u_k u_q + 2f_0^*\tilde{v}_Q u_q \tilde{v}_k^* - f_0\tilde{v}_Q\tilde{v}_k^*\tilde{v}_q^* - 2f_0 u_Q^* u_k \tilde{v}_q^*]\mathrm{e}^{-2|z|^2}\hat{a}_Q^\dagger \hat{a}_q \hat{a}_k$$

$$\gamma \sim \frac{k^4}{2^9 3^3 \pi n}$$



$$\epsilon(k_x, k_y)$$

•Density matrix

•**Density-phase representation**:

$$\hat{\Psi}(z) = (n(z))^{1/2} \exp(i\hat{\Phi})$$

$$\Phi(z) = (4n(z))^{-1/2} \mathrm{e}^{-|z|^2/2} \sum_k (u_k(z) + v_k(z)) \hat{a}_k \mathrm{e}^{-i\epsilon_k t} + h.c.$$

$$\varphi(r) = < \Phi(\mathbf{r})\Phi(0) > \sim \sum_{\mathbf{k}} \frac{K_0}{n\epsilon_{\mathbf{k}} L^2} \exp[i\mathbf{k}\mathbf{r}] = \frac{K_0}{c_\epsilon n \beta} \int \frac{\exp[i\mathbf{k}\mathbf{r}]}{k^2} \frac{d^2k}{(2\pi)^2}$$

$$< \Psi(z) > = (n)^{-1/2} \exp[-\varphi(0)^2] \rightarrow 0$$

•*Density matrix:*

$$\rho(|z_1 - z_2|) =< \hat{\Psi}(z_1)\hat{\Psi}^\dagger(z_2) >$$

$$\rho(\mathbf{r}) \sim \exp[\varphi(\mathbf{r}) - \varphi(0)]$$

$$\varphi(\mathbf{r}) - \varphi(0) = \frac{K_0}{c_\epsilon n\beta} \int \frac{\exp[i\mathbf{k}\mathbf{r}] - 1}{k^2} \frac{d^2k}{(2\pi)^2}$$

$$\rho(r) \sim \sqrt{n(r)\, n(0)} \left(\frac{l}{r}\right)^\eta$$

$$\eta = \frac{K_0}{2\pi c_\epsilon n\beta}$$

→ •**The algebraic decay of the density matrix for T =0!**

RESULTS

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