Canonical conservation laws and integrability conditions for difference equations

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- Difference equation $Q = 0$ and Dynamical variables.

- Difference fields $\mathcal{F}_Q, \mathcal{F}_0, \mathcal{F}_{\mathrm{s}}, \mathcal{F}_{\mathbf{t}}$, the elimination map.

- Symmetries and conservation laws

- Recursion operators for difference equations

- Formal difference series, difference Adler Theorem.

- Canonical conservation laws: integrability conditions

Difference equations on $\mathbb{Z}^2$ can be seen as a discrete analogue of partial differential equations with two independent variables.

Let us denote by $u = u(n, m)$ a complex-valued function $u : \mathbb{Z}^2 \mapsto \mathbb{C}$ where n and m are "independent variables" and u will play the rôle of a "dependent" variable in a difference equation.

Instead of partial derivatives we have two commuting shift maps $\mathcal{S}$ and $\mathcal{T}$ defined as

$$\mathcal{S} : u \mapsto u_{1,0} = u(n+1, m), \qquad \mathcal{T} : u \mapsto u_{0,1} = u(n, m+1)$$

For uniformity of notations, we denote u as $u_{0,0}$.

In the theory of difference equations we shall treat symbols $u_{p,q}$ as **variables**.

We denote $U = \{u_{p,q} \,|\, (p,q) \in \mathbb{Z}^2\}$.

For a function $f = f(u_{p_1,q_1}, \ldots, u_{p_k,q_k})$

$$\mathcal{S}^i \mathcal{T}^j (f) = f_{i,j} = f(u_{p_1+i,q_1+j}, \ldots, u_{p_k+i,q_k+j}).$$

A quadrilateral difference equation can be defined as

$$Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0 \ ,$$

where $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1})$ is an **irreducible polynomial** of the "dependent variable" $u = u_{0,0}$ and its shifts.

$$Q_{p,q} = Q(u_{p,q}, u_{p+1,q}, u_{p,q+1}, u_{p+1,q+1}) = 0, \qquad (p,\ q) \in \mathbb{Z}^2 \ .$$

We shall assume that Q is an irreducible **affine-linear polynomial** which depends non-trivially on all variables:

$$\frac{\partial Q}{\partial u_{i,j}} \neq 0, \quad \frac{\partial^2 Q}{\partial^2 u_{i,j}} = 0, \qquad\qquad i, j \in \{0, 1\}.$$

Let $\mathbb{C}[U]$ be the ring of polynomials.

$\mathcal{S}, \mathcal{T} \in \operatorname{Aut} \mathbb{C}[U]$ and thus $\mathbb{C}[U]$ is a **difference ring**.

The **difference ideal**

$$J_Q = \langle \{ Q_{p,q} \,|\, (p,\ q) \in \mathbb{Z}^2 \} \rangle$$

is prime and thus the quotient ring $\mathbb{C}[U]/J_Q$ is an integral domain.

Solutions of the difference equation are points of the affine variety $V(J_Q)$.

Rational functions of variables $u_{p,q}$ on $V(J_Q)$ form a field

$$\mathcal{F}_Q = \{[a]/[b] \,|\, a, b \in \mathbb{C}[U],\ b \notin J_Q\},$$

where $[a]$ denotes the class of equivalent polynomials (two polynomials $f, g \in \mathbb{C}[U]$ are equivalent, denoted by $f \equiv g$, if $f - g \in J_Q$).

For $a, b, c, d \in \mathbb{C}[U],\ b, d \notin J_Q$, rational functions a/b and c/d represent the same element of $\mathcal{F}_Q$ if $ad - bc \in J_Q$.

The fields of rational functions of variables

$$U_{\mathrm{s}} = \{u_{n,0} \,|\, n \in \mathbb{Z}\}, \quad U_{\mathbf{t}} = \{u_{0,n} \,|\, n \in \mathbb{Z}\}, \quad U_0 = U_{\mathrm{s}} \cup U_{\mathbf{t}}.$$

are denoted respectively as

$$\mathcal{F}_{\mathrm{s}} = \mathbb{C}(U_{\mathrm{s}}), \qquad \mathcal{F}_{\mathbf{t}} = \mathbb{C}(U_{\mathbf{t}}), \qquad \mathcal{F}_0 = \mathbb{C}(U_0).$$

In the affine-linear case we can uniquely resolve equation $Q = 0$ with respect to each variable

$$u_{0,0} = F(u_{1,0}, u_{0,1}, u_{1,1}), \qquad u_{1,0} = G(u_{0,0}, u_{0,1}, u_{1,1}),$$
$$u_{0,1} = H(u_{0,0}, u_{1,0}, u_{1,1}), \qquad u_{1,1} = M(u_{0,0}, u_{1,0}, u_{0,1}).$$

We can recursively and uniquely express any variable $u_{p,q}$ in terms of the variables $U_0 = U_{\mathbf{s}} \cup U_{\mathbf{t}}$.

Definition 1. *For elements of U the* **elimination map** $\mathcal{E} : U \mapsto \mathbb{C}(U_0)$ *is defined recursively:*

$$
\begin{array}{ll}
\forall p \in \mathbb{Z}, & \mathcal{E}(u_{0,p}) = u_{0,p}, \qquad \mathcal{E}(u_{p,0}) = u_{p,0}\,, \\
\text{if } p > 0, q > 0, & \mathcal{E}(u_{p,q}) = M(\mathcal{E}(u_{p-1,q-1}), \mathcal{E}(u_{p,q-1}), \mathcal{E}(u_{p-1,q}))\,, \\
\text{if } p < 0, q > 0, & \mathcal{E}(u_{p,q}) = H(\mathcal{E}(u_{p,q-1}), \mathcal{E}(u_{p+1,q-1}), \mathcal{E}(u_{p+1,q}))\,, \\
\text{if } p > 0, q < 0, & \mathcal{E}(u_{p,q}) = G(\mathcal{E}(u_{p-1,q}), \mathcal{E}(u_{p-1,q+1}), \mathcal{E}(u_{p,q+1}))\,, \\
\text{if } p < 0, q < 0, & \mathcal{E}(u_{p,q}) = F(\mathcal{E}(u_{p+1,q}), \mathcal{E}(u_{p,q+1}), \mathcal{E}(u_{p+1,q+1}))\,.
\end{array}
$$

For polynomials $f(u_{p_1,q_1},\ldots,u_{p_k,q_k}) \in \mathbb{C}[U]$ *the elimination map* $\mathcal{E} : \mathbb{C}[U] \mapsto \mathbb{C}(U_0)$ *is defined as*

$$\mathcal{E} : f(u_{p_1,q_1},\ldots,u_{p_k,q_k}) \mapsto f(\mathcal{E}(u_{p_1,q_1}),\ldots,\mathcal{E}(u_{p_k,q_k})) \in \mathbb{C}(U_0).$$

For rational functions $a/b,\ a,b \in \mathbb{C}[U],\ b \notin J_Q$ *the elimination map* $\mathcal{E}$ *is defined as*

$$\mathcal{E} : a/b \mapsto \mathcal{E}(a)/\mathcal{E}(b).$$

Variables U_0 we shall call the **dynamical** variables.

$\mathcal{E}(u_{p,q})$ is a rational function of $|p|+|q|+1$ dynamical variables ($pq \neq 0$)

$$\mathcal{E}(u_{p,q}) \in \mathbb{C}(\{u_{n,0}, u_{0,m} \,|\, 0 \le |n-p| \le |p|,\ 0 \le |m-q| \le |q|\}).$$

The elimination map $\mathcal{E} : \mathbb{C}[U] \mapsto \mathbb{C}(U_0)$ is a **difference ring homomorphism**

$$\mathrm{Ker}\,\mathcal{E} = J_Q\,, \qquad \mathrm{Im}\,\mathcal{E} \sim \mathbb{C}[U]/J_Q.$$

The field $\mathbb{C}(U_0)$ is a **difference field** with automorphisms $\mathcal{E} \circ \mathcal{S}$ and $\mathcal{E} \circ \mathcal{T}$. The map $\mathcal{E} : \mathcal{F}_Q \mapsto \mathbb{C}(U_0)$ is a **difference field isomorphism**.

Map $\mathcal{E}$ is a useful tool to establish whether two rational functions f, g of variables U are equivalent (i.e. represent the same element of $\mathcal{F}_Q$): $f \equiv g \Leftrightarrow \mathcal{E}(f) = \mathcal{E}(g)$.

Symmetries and conservation laws of difference equations

Definition 2. *Let $Q = 0$ be a difference equation. Then $K \in \mathcal{F}_Q$ is called a* **symmetry** *(a generator of an infinitesimal symmetry) of the difference equation if*

$$D_Q(K) \equiv 0.$$

Here D_Q is the Frechét derivative of Q defined as

$$D_Q = \sum_{i,j} Q_{u_{i,j}} \mathcal{S}^i \mathcal{T}^j \ , \qquad Q_{u_{i,j}} = \frac{\partial Q}{\partial u_{i,j}}.$$

What one has to check is that $\mathcal{E}(D_Q(K)) = 0$.

If K is a symmetry and $u = u(n,m)$ is a solution of a difference equation $Q = 0$, then the infinitesimal transformation of solution u:

$$\widehat{u} = u + \epsilon K$$

satisfies equation

$$Q(\widehat{u}_{0,0}, \widehat{u}_{1,0}, \widehat{u}_{0,1}, \widehat{u}_{1,1}) \equiv \mathcal{O}(\epsilon^2).$$

If the difference equation $Q = 0$ admits symmetries, then they form a Lie algebra. With a symmetry $K \in \mathcal{F}_Q$ we associate an evolutionary derivation ($\mathcal{S} X_K = X_K \mathcal{S}$, $\mathcal{T} X_K = X_K \mathcal{T}$) of $\mathcal{F}_Q$ (or a vector field on $\mathcal{F}_Q$):

$$X_K = \sum_{(p,q)\in\mathbb{Z}^2} K_{p,q} \frac{\partial}{\partial u_{p,q}}, \qquad K_{p,q} = \mathcal{S}^p \mathcal{T}^q (K)$$

For any $a \in J_Q$ we have $X_K(a) \in J_Q$ and thus the evolutionary derivation X_K is defined **correctly** on $\mathcal{F}_Q$.

$$X_F X_G - X_G X_F = X_H,$$

where $H = [F, G]$ is also a symmetry, with $[F, G]$ denoting the **Lie bracket**

$$[F, G] = X_F(G) - X_G(F) = D_G(F) - D_F(G) \in \mathcal{F}_Q.$$

The Lie algebra of symmetries of the difference equation $Q = 0$ will be denoted as $\mathfrak{A}_Q$.

Existence of an infinite dimensional Lie algebra $\mathfrak{A}_Q$ is a characteristic property of integrable equations and can be taken as a **definition of integrability**.

Symmetry K can be represented by $\mathcal{E}(K) \in \mathcal{F}_0$.

It is known, that for a quadrilateral equation a symmetry is a sum of two functions

$$K = K_{\mathbf{s}}(u_{N_1,0}, \ldots, u_{N_2,0}) + K_{\mathbf{t}}(u_{0,M_1}, \ldots, u_{0,M_2}) \ .$$

For the ABS equations any "five point symmetry" $K = K(u_{0,0}, u_{-1,0}, u_{0,-1}, u_{1,0}, u_{0,1})$ is a **sum of symmetries**

$$K = K_{\mathbf{s}}(u_{-1,0}, u_{0,0}, u_{1,0}) + K_{\mathbf{t}}(u_{0,-1}, u_{0,0}, u_{0,1}).$$

For $K_{\mathbf{s}} \in \mathcal{F}_{\mathbf{s}}$: $\operatorname{ord} K_{\mathbf{s}}(u_{N_1,0}, \ldots, u_{N_2,0}) = (N_1, N_2)$.
For $K_{\mathbf{t}} \in \mathcal{F}_{\mathbf{t}}$: $\operatorname{ord} K_{\mathbf{t}}(u_{0,M_1}, \ldots, u_{0,M_2}) = (M_1, M_2)$.

Definition 3. *(1) A pair* $(\rho, \sigma) \in \hat{\mathcal{F}}_Q$ *is called a* **conservation law** *for the difference equation* $Q = 0$*, if*

$$(\mathcal{T} - \mathbf{1})(\rho) \equiv (\mathcal{S} - \mathbf{1})(\sigma).$$

Functions ρ *and* σ *will be referred to as the density and the flux of the conservation law and* $\mathbf{1}$ *denotes the identity map.*

(2) A conservation law is called **trivial**, *if functions* ρ *and* σ *are components of a (difference) gradient of some element* $H \in \mathcal{F}_Q$*, i.e.*

$$\rho = (\mathcal{S} - \mathbf{1})(H), \qquad \sigma = (\mathcal{T} - \mathbf{1})(H).$$

(3) If $\rho_1 - \rho_2 \in \mathrm{Im}(\mathcal{S} - \mathbf{1})$*, then* $\rho_1 \cong \rho_2$*.*

Typically conserved densities belong to $\mathcal{F}_\mathrm{s}$ or $\mathcal{F}_\mathbf{t}$.

Euler's operator gives a criteria to determine whether two elements of $\mathcal{F}_\mathrm{s}$ are equivalent or not.

Definition 4. *Let $f \in \mathcal{F}_\mathrm{s}$ has order (N_1, N_2), then the variational derivative δ_s of f is defined as*

$$\delta_\mathrm{s}(f) = \sum_{k=N_1}^{N_2} \mathcal{S}^{-k}\left(\frac{\partial f}{\partial u_{k,0}}\right).$$

For $\rho, \varrho \in \mathcal{F}_\mathrm{s}$, $\rho \overset{\smile}{=} \varrho \;\Leftrightarrow\; \delta_\mathrm{s}(\rho) = \delta_\mathrm{s}(\varrho)$.
If ρ is trivial then $\delta_\mathrm{s}(\rho) = 0$.

The **order of a density** $\rho \in \mathcal{F}_\mathrm{s}$ is defined as $\mathrm{ord}_{\delta_\mathrm{s}}(\rho) = N_2 - N_1$, where $(N_1, N_2) = \mathrm{ord}(\delta_\mathrm{s}(\rho))$.

Recursion operators for difference equations

Definition 5. *(1) Elements of* $\mathcal{F}_Q[\mathcal{S}]$ *are called* $\mathbf{s}$*-difference operators.*
(2) Elements of $\mathcal{F}_Q(\mathcal{S})$ *are called* $\mathbf{s}$*-pseudo-difference operators.*

Similarly one can define $\mathbf{t}$-difference and $\mathbf{t}$-pseudo-difference operators.

The action of a difference operator $A \in \mathcal{F}_Q[\mathcal{S}]$ on elements of $\mathcal{F}_Q$ is naturally defined and $\mathrm{Dom}(A) = \mathcal{F}_Q$.

The domain of a pseudo-difference operator $B \in \mathcal{F}_Q(\mathcal{S})$ is defined as $\mathrm{Dom}(B) = \{a \in \mathcal{F}_Q \,|\, B(a) \in \mathcal{F}_Q\}$.
For instance, if $B = FG^{-1}$ where $F, G \in \mathcal{F}_Q[\mathcal{S}]$, then $\mathrm{Dom}(B) = \mathrm{Im}\, G$.

A **recursion operator** of a **difference equation** $Q = 0$ is a pseudo-difference operator $\mathfrak{R}$ such that

$$\mathfrak{R} : \mathrm{Dom}(\mathfrak{R}) \cap \mathfrak{A}_Q \mapsto \mathfrak{A}_Q,$$

where $\mathfrak{A}_Q$ is the linear space of symmetries of this difference equation.

In other words, if the action of $\mathfrak{R}$ on a symmetry $K \in \mathcal{F}_Q$ is defined, i.e. $\mathfrak{R}(K) \in \mathcal{F}_Q$, then $\mathfrak{R}(K)$ is a symmetry of the same difference equation.

Theorem 1. *Let* $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0$ *be a difference equation.*

(i) *If there exist two* s*–pseudo-differential operators* $\mathfrak{R}$ *and* $\mathfrak{P}$ *such that*

$$D_Q \circ \mathfrak{R} = \mathfrak{P} \circ D_Q,$$

then $\mathfrak{R}$ *is a recursion operator of the difference equation.*

(ii) *Relation (1) is valid if and only if*

$$\mathcal{T}(\mathfrak{R}) - \mathfrak{R} = [\Phi \circ \mathfrak{R}, \Phi^{-1}] \ ,$$

where $\Phi = (Q_{u_{1,1}}\mathcal{S} + Q_{u_{0,1}})^{-1} \circ (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}})$*, and the operator* $\mathfrak{P}$ *satisfies*

$$\mathfrak{P} = (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}}) \circ \mathfrak{R} \circ (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}})^{-1}.$$

Theorem 2. *Let* $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0$ *be a difference equation.*

(i) *If there exist two* $\mathfrak{t}$*–pseudo-differential operators* $\widehat{\mathfrak{R}}$ *and* $\widehat{\mathfrak{P}}$ *such that*

$$D_Q \circ \widehat{\mathfrak{R}} = \widehat{\mathfrak{P}} \circ D_Q, \tag{1}$$

then $\widehat{\mathfrak{R}}$ *is a recursion operator of the difference equation.*

(ii) *Relation (1) is valid if and only if*

$$\mathcal{S}(\widehat{\mathfrak{R}}) - \widehat{\mathfrak{R}} = [\Psi \circ \widehat{\mathfrak{R}}, \Psi^{-1}]$$

where $\Psi = (Q_{u_{1,1}}\mathcal{T} + Q_{u_{1,0}})^{-1} \circ (Q_{u_{0,1}}\mathcal{T} + Q_{u_{0,0}})$*, and the operator* $\widehat{\mathfrak{P}}$ *can be written as*

$$\widehat{\mathfrak{P}} = (Q_{u_{0,1}}\mathcal{T} + Q_{u_{0,0}}) \circ \widehat{\mathfrak{R}} \circ (Q_{u_{0,1}}\mathcal{T} + Q_{u_{0,0}})^{-1}.$$

Corollary 1. *1. Under the conditions of Theorem 1, the pseudo-difference operator* $\mathfrak{R}$ *satisfies the following equations*

$$\mathcal{T}(\mathfrak{R}^n) - \mathfrak{R}^n = [\Phi \circ \mathfrak{R}^n, \Phi^{-1}], \qquad n \in \mathbb{Z}, \tag{2}$$

2. Under the conditions of Theorem 2, the pseudo-difference operator $\hat{\mathfrak{R}}$ *satisfies equations*

$$\mathcal{S}(\hat{\mathfrak{R}}^n) - \hat{\mathfrak{R}}^n = [\Psi \circ \hat{\mathfrak{R}}^n, \Psi^{-1}] \qquad n \in \mathbb{Z},$$

Proof. It that $D_Q \circ \mathfrak{R}^n = \mathfrak{P}^n \circ D_Q, n \in \mathbb{Z}$. Thus, we can apply Theorem 1 to $\mathfrak{R}^n$ and $\mathfrak{P}^n$ to produce (2). The proof of the second part of the Corollary is similar. ∎

Formal difference series, difference Adler Theorem.

Definition 6. *A formal Laurent series of order N is defined as a formal semi-infinite sum*

$$A = a_N \mathcal{S}^N + a_{N-1} \mathcal{S}^{N-1} + \cdots + a_1 \mathcal{S} + a_0 + a_{-1} \mathcal{S}^{-1} + \cdots \,,$$

where $a_k \in \mathcal{F}_Q, \quad a_N \neq 0, \quad N \in \mathbb{Z}$.

A formal Taylor series of order $-N$ is defined as a formal semi-infinite sum

$$C = c_{-N} \mathcal{S}^{-N} + c_{1-N} \mathcal{S}^{1-N} + \cdots + c_{-1} \mathcal{S}^{-1} + c_0 + c_1 \mathcal{S} + \cdots \,,$$

where $c_k \in \mathcal{F}_Q, \quad c_{-N} \neq 0, \quad N \in \mathbb{Z}$.

Laurent formal series form a skew-field. Sums and products (compositions) of formal series are formal series. The product is associative, but not commutative.

For any formal series A there exists a formal series A^{-1} such that $A \circ A^{-1} = A^{-1} \circ A = \mathbf{1}$. In order to find the first n coefficients of A^{-1} one needs to know exactly the first n coefficients of A.

Any pseudo-difference operator B can be uniquely represented by a Laurent formal series B_L. For example for $B = (a\mathcal{S} + b)^{-1}$ we have

$$B_L = \alpha_{-1}\mathcal{S}^{-1} + \alpha_{-2}\mathcal{S}^{-2} + \alpha_{-3}\mathcal{S}^{-3} + \cdots ,$$

where the coefficients $\alpha_k \in \mathcal{F}_Q$ can be found recursively:

$$\alpha_{-1} = \mathcal{S}^{-1}\left(\frac{1}{a}\right),\ \alpha_{-n} = -\mathcal{S}^{-1}\left(\frac{\alpha_{1-n}b}{a}\right).$$

Definition 7. *Let A_L denote the Laurent series representations of a pseudo-differential operator A. Then* ordA_L *is called the* **Laurent order** *of A.*

Definition 8. *Let A be a formal series of order N*

$$A = a_N\mathcal{S}^N + a_{N-1}\mathcal{S}^{N-1} + \cdots + a_1\mathcal{S} + a_0 + a_{-1}\mathcal{S}^{-1} + \cdots ,$$

The **residue** $\operatorname{res}(A)$ *and* **logarithmic residue** $\operatorname{res}\ln(A)$ *are defined as*

$$\operatorname{res}(A) = a_0, \qquad \operatorname{res}\ln(A) = \ln(a_N)\,.$$

Theorem 3. *Let $A = a_N\mathcal{S}^N + a_{N-1}\mathcal{S}^{N-1}\cdots$ and $B = b_M\mathcal{S}^M + b_{M-1}\mathcal{S}^{M-1}\cdots$ be two Laurent formal series of order N and M respectively. Then*

$$\operatorname{res}[A, B] = (\mathcal{S} - \mathbf{1})(\sigma(A, B)),$$

where $\sigma(A, B) \in \mathcal{F}_Q$

$$\sigma(A, B) = \sum_{n=1}^{N}\sum_{k=1}^{n}\mathcal{S}^{-k}(a_{-n})\mathcal{S}^{n-k}(b_n) - \sum_{n=1}^{M}\sum_{k=1}^{n}\mathcal{S}^{-k}(b_{-n})\mathcal{S}^{n-k}(a_n).$$

Canonical conservation laws: integrability conditions

Theorem 4. *If a difference equation possesses a recursion operator* $\mathfrak{R}$, $\text{ord}_L(\mathfrak{R}) = N > 0$*, then it has infinitely many* canonical *conservation laws*

$$(\mathcal{T} - 1)\rho_{nN} = (\mathcal{S} - 1)\sigma_{nN}, \quad n = 0, 1, 2, \ldots$$

with **canonical conserved densities**

$$\rho_0 = \text{res} \ln \mathfrak{R}_L, \quad \rho_{nN} = \text{res}\mathfrak{R}_L^n, \; n > 0,$$

and the corresponding **canonical fluxes**

$$\sigma_0 = \sum_{k=0}^{N-1} \mathcal{S}^{k-1}(\ln \alpha_0), \quad \sigma_{nN} = \sigma(\Phi_L \circ \mathfrak{R}_L^n, \Phi_L^{-1})$$

where α_0 *is the first coefficient in the Laurent expansion of* Φ*:* $\alpha_0 = \mathcal{S}^{-1}\left(\frac{Q_{u_{1,0}}}{Q_{u_{1,1}}}\right)$ *and* $\sigma(A, B)$ *is defined in Theorem 3.*

If a recursion operator is known, then Theorem 4 gives us a completely algorithmic way to find explicitly a sequence of conservation laws, including both the densities ρ_k and the corresponding fluxes σ_k. The residues of powers a formal series are easy to compute. For instance, if

$$\mathfrak{R} = r_1\mathcal{S} + r_0 + r_{-1}\mathcal{S}^{-1} + r_{-2}\mathcal{S}^{-2} + r_{-3}\mathcal{S}^{-3} +$$

then

$$\begin{aligned}
&\mathrm{res}\ln\mathfrak{R} = \ln r_1, \quad \mathrm{res}\mathfrak{R} = r_0,\\
&\mathrm{res}\mathfrak{R}^2 = \mathcal{S}^{-1}(r_1)r_{-1} + r_0^2 + r_1\mathcal{S}^{-1}(r_{-1}),\\
&\mathrm{res}\mathfrak{R}^3 = \mathcal{S}^{-2}(r_1)\mathcal{S}^{-1}(r_1)r_{-2} +\\
&+\mathcal{S}^{-1}(r_0)\mathcal{S}^{-1}(r_1)r_{-1} + 2\mathcal{S}^{-1}(r_1)r_{-1}r_0 + \mathcal{S}^{-1}(r_1)r_1\mathcal{S}(r_{-2}) +\\
&+r_0^3 + 2r_0r_1\mathcal{S}(r_{-2}) + r_1\mathcal{S}(r_{-1})\mathcal{S}(r_0) + r_1\mathcal{S}(r_1)\mathcal{S}^2(r_{-2}).
\end{aligned}$$

Proposition 1. *If a recursion operator* $\mathfrak{R}$ *is represented by a first order formal series* $\mathfrak{R}_L = r_1\mathcal{S} + r_0 + r_{-1}\mathcal{S}^{-1} + \cdots$, *then*

$$\text{(i)} \qquad (\mathcal{T} - \mathbf{1})(\ln r_1) = (\mathcal{S} - \mathbf{1})\mathcal{S}^{-1}\left(\ln \frac{Q_{u_{1,1}}}{Q_{u_{1,0}}}\right),$$

$$\text{(ii)} \qquad (\mathcal{T} - \mathbf{1})(r_0) = (\mathcal{S} - \mathbf{1})\mathcal{S}^{-1}(r_1 F),$$

$$\text{(iii)} \qquad (\mathcal{T} - \mathbf{1})(r_{-1}\mathcal{S}^{-1}(r_1) + r_0^2 + r_1\mathcal{S}(r_{-1})) = (\mathcal{S} - \mathbf{1})(\sigma_2),$$

where

$$\begin{aligned} \sigma_2 \;=\; & \mathcal{S}^{-1}(r_1\, F)\left\{\mathcal{S}^{-1}(r_0) + r_0 - \mathcal{S}^{-2}(r_1\, F)\right\} - \\ & -(1 + \mathcal{S}^{-1})\left(r_1\, G\, \mathcal{S}^{-1}(r_1\, F)\right), \end{aligned}$$

and F, G *denote*

$$F = \frac{Q_{u_{0,1}}\mathcal{S}^{-1}(Q_{u_{1,0}}) - Q_{u_{0,0}}\mathcal{S}^{-1}(Q_{u_{1,1}})}{Q_{u_{1,0}}\mathcal{S}^{-1}(Q_{u_{1,1}})}, \qquad G = \frac{Q_{u_{0,0}}}{Q_{u_{1,0}}}.$$

Proposition 2. *If a recursion operator* $\widehat{\mathfrak{R}}$ *is represented by a first order formal Laurent* $\mathcal{T}$ *series* $\widehat{\mathfrak{R}}_L = \widehat{r}_1\mathcal{T} + \widehat{r}_0 + \widehat{r}_{-1}\mathcal{T}^{-1} + \cdots$, *then*

$$\text{(i)} \qquad (\mathcal{S}-\mathbf{1})(\ln \widehat{r}_1) = (\mathcal{T}-\mathbf{1})\mathcal{T}^{-1}\left(\ln \frac{Q_{u_{1,1}}}{Q_{u_{0,1}}}\right),$$

$$\text{(ii)} \qquad (\mathcal{S}-\mathbf{1})(\widehat{r}_0) = (\mathcal{T}-\mathbf{1})\mathcal{T}^{-1}\left(\widehat{r}_1\widehat{F}\right),$$

$$\text{(iii)} \qquad (\mathcal{S}-\mathbf{1})(\widehat{r}_{-1}\mathcal{S}^{-1}(\widehat{r}_1) + \widehat{r}_0^2 + \widehat{r}_1\mathcal{S}(\widehat{r}_{-1})) = (\mathcal{T}-\mathbf{1})(\widehat{\sigma}_2),$$

where

$$\begin{aligned} \widehat{\sigma}_2 \; &= \; \mathcal{S}^{-1}(\widehat{r}_1\,\widehat{F})\left\{\mathcal{S}^{-1}(\widehat{r}_0) + \widehat{r}_0 - \mathcal{S}^{-2}\left(\widehat{r}_1\,\widehat{F}\right)\right\} - \\ &- \; (1+\mathcal{S}^{-1})\left(\widehat{r}_1\,\widehat{G}\,\mathcal{S}^{-1}\left(\widehat{r}_1\,\widehat{F}\right)\right), \end{aligned}$$

and $\widehat{F}, \widehat{G}$ *denote*

$$\widehat{F} = \frac{Q_{u_{1,0}}\mathcal{T}^{-1}(Q_{u_{0,1}}) - Q_{u_{0,0}}\mathcal{T}^{-1}(Q_{u_{1,1}})}{Q_{u_{0,1}}\mathcal{T}^{-1}(Q_{u_{1,1}})}, \qquad \widehat{G} = \frac{Q_{u_{0,0}}}{Q_{u_{0,1}}}.$$

The Viallet equation (Q5)

$$
\begin{aligned}
Q \; := \; & a_1 u_{0,0} u_{1,0} u_{0,1} u_{1,1} + a_2 (u_{0,0} u_{1,0} u_{0,1} + u_{1,0} u_{0,1} u_{1,1} \\
& \quad + u_{0,1} u_{1,1} u_{0,0} + u_{1,1} u_{0,0} u_{1,0}) \\
& + a_3 (u_{0,0} u_{1,0} + u_{0,1} u_{1,1}) + a_4 (u_{1,0} u_{0,1} + u_{0,0} u_{1,1}) \\
& + a_5 (u_{0,0} u_{0,1} + u_{1,0} u_{1,1}) \\
& + a_6 (u_{0,0} + u_{1,0} + u_{0,1} + u_{1,1}) + a_7 \\
& = 0 \, ,
\end{aligned}
$$

where a_i are free complex parameters.

- All of the ABS (**Adler, Bobenko, Suris**) equations can be obtained from $\mathbf{Q_5}$;
- For generic coefficients it can be transformed into Adler's equation ($\mathbf{Q_4}$);
- $\mathbf{Q_5}$ is invariant under $u_{1,0} \leftrightarrow u_{0,1}$ and $a_3 \leftrightarrow a_5$.

Genralised symmetries K: $D_Q[K]=0$

- Tongas-Tsoubelis-Xenitidis; Rasin-Hydon: the ABS equations (mastersymmetries);

- A generalised symmetry of order $(-1,1)$ for the Viallet equation (Tongas-Tsoubelis-Xenitidis):

  $$K^{(1)} := \frac{h}{u_{1,0}-u_{-1,0}} - \frac{1}{2}\partial_{u_{1,0}} h = \frac{h_{-1}}{u_{1,0}-u_{-1,0}} + \frac{1}{2}\partial_{u_{-1,0}} h_{-1}\,,$$

  where $h(u_{0,0},u_{1,0}) = Q\,\partial_{u_{0,1}}\partial_{u_{1,1}}Q \,-\, \partial_{u_{0,1}}Q\,\partial_{u_{1,1}}Q.$

- **Statement**: The Viallet equation possesses a symmetry of order $(-2,2)$

  $$K^{(2)} = \frac{h\,h_{-1}}{(u_{1,0}-u_{-1,0})^2}\left(\frac{1}{u_{2,0}-u_{0,0}} + \frac{1}{u_{0,0}-u_{-2,0}}\right)\,.$$

Recursion operator of the Viallet equation

Theorem. The Viallet equation possesses a recursion operator $\mathcal{R} = \mathcal{H} \circ \mathcal{I}$.

$$\mathcal{I} = \frac{1}{h}\mathcal{S} - \mathcal{S}^{-1}\frac{1}{h} \text{ ; (symplectic operator)}$$

$$\mathcal{H} = \frac{h_{-1}\, h\, h_1}{(u_{1,0} - u_{-1,0})^2 (u_{2,0} - u_{0,0})^2}\,\mathcal{S}$$

$$- \mathcal{S}^{-1}\frac{h_{-1}\, h\, h_1}{(u_{1,0} - u_{-1,0})^2 (u_{2,0} - u_{0,0})^2}$$

$$+ 2\, K^{(1)}\, \mathcal{S}\, (\mathcal{S} - \mathbf{1})^{-1} K^{(2)} + 2\, K^{(2)}\, (\mathcal{S} - \mathbf{1})^{-1} K^{(1)}$$

(Hamiltonian operator) .

Symplectic and Hamiltonian Operators

$$\mathcal{I} : \mathfrak{h} \to \Omega^1 \Longrightarrow \omega = \frac{1}{2} \int \mathrm{d}u \wedge \mathcal{I} \mathrm{d}u$$

if 2-form ω is anti-symmetric and is closed, we say $\mathcal{I}$ is **symplectic**.

$$\mathcal{H} : \Omega^1 \to \mathfrak{h} \Longrightarrow \left\{ \int f, \ \int g \right\} = < \delta(f), \mathcal{H}\delta(g) >$$

if the Poisson bracket defined is anti-symmetric and satisfies the Jacobi identity, we say $\mathcal{H}$ is **Hamiltonian**.

(cf. Dorfman, 1993 & Olver 1993)

Symplectic and Hamiltonian Operators of Q5

Symplectic: symmetries $\mapsto$ covariants
Hamiltonian: covariants $\mapsto$ symmetries

$$\mathcal{I}(K^{(1)}) = \frac{1}{u_{2,0}-u_{0,0}} - \frac{1}{u_{0,0}-u_{-2,0}} + \frac{1}{2}\frac{\partial_{u_{0,0}} h}{h} + \frac{1}{2}\frac{\partial_{u_{0,0}} h_{-1}}{h_{-1}}$$
$$= \delta_{\mathrm{s}}\left(\ln h_{-1} - 2\ln(u_{1,0}-u_{-1,0})\right)$$

$$\mathcal{H}\mathcal{I}(K^{(1)}) = \frac{h\,h_{-1}}{(u_{1,0}-u_{-1,0})^2}\left(\frac{K_2^{(1)}}{(u_{2,0}-u_{0,0})^2} + \frac{K_{-2}^{(1)}}{(u_{0,0}-u_{-2,0})^2}\right.$$
$$\left. + \left(\frac{1}{u_{2,0}-u_{0,0}} + \frac{1}{u_{0,0}-u_{-2,0}}\right) K^{(1)} K^{(2)}\right)$$