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# Internal waves in a compressible two-layer atmospheric model: The Hamiltonian description

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Slow flows of an ideal compressible fluid (gas) in the gravity field in the presence of two isentropic layers are considered, with a small difference of specific entropy between them. Assuming irrotational flows in each layer and neglecting acoustic degrees of freedom, we derive Hamiltonian equations of motion for the interface. The idealized system under consideration is the simplest theoretical model for studying internal waves in a sharply stratified atmosphere, where decrease of mean equilibrium gas density $\bar{\rho}(z)$ with altitude due to compressibility is essentially taken into account. For planar flows, a generalization is made to the case when in each layer there is a constant potential vorticity. Investigated in more details is the system with a model dependence $\bar{\rho}(z) \propto \exp(-2\alpha z)$, for which the Hamiltonian can be expressed explicitly. A long-wave regime is considered, and an approximate weakly nonlinear equation of the form $u_t + auu_x - b[-\hat{\partial}_x^2 + \alpha^2]^{1/2} u_x = 0$ (known as Smith's equation) is derived for evolution of a unidirectional wave.

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## Approximate equations and their Hamiltonian structure

In each layer we have non-stationary Bernoulli equation and continuity equation,

$$\partial_t \varphi + \frac{(\nabla\varphi)^2}{2} = -w(\rho) - gz + const, \tag{1}$$

$$\partial_t \rho + \nabla \cdot (\rho \nabla \varphi) = 0, \tag{2}$$

where $\varphi(\mathbf{r}, t)$ is the potential for the velocity field $\mathbf{v}$, $\rho(\mathbf{r}, t)$ is the gas density, $w(\rho)$ is the specific enthalpy which is defined by formula

$$w(\rho) = w_{1,2}(\rho) = \int_0^\rho \frac{dp_{1,2}(\rho)}{\rho}. \tag{3}$$

In the equilibrium state the velocity potential $\varphi = 0$, the enthalpy $w_{1,2}(\bar{\rho}_{1,2}(z)) = const_{1,2} - gz$, and the pressure is related to the density by the hydrostatic formula

$$\bar{p}_{1,2}(z) = p_0 - g \int_h^z \bar{\rho}_{1,2}(z) dz. \tag{4}$$

We consider slow flows when

$$p_{1,2} = \bar{p}_{1,2}(z) + \tilde{p}_{1,2},$$

$$w_{1,2} \approx const_{1,2} - gz + \tilde{p}_{1,2}/\bar{\rho}(z),$$

where $\tilde{p}_{1,2}$ are relatively small corrections to the pressure field due to the fluid flow. Equations of slow motion in the main order in $v/c$ take the form

$$\partial_t \varphi_{1,2} + \frac{(\nabla \varphi_{1,2})^2}{2} + \frac{\tilde{p}_{1,2}}{\bar{\rho}(z)} \;=\; 0, \qquad (5)$$

$$\nabla \cdot (\bar{\rho}(z) \nabla \varphi_{1,2}) \;=\; 0. \qquad (6)$$

It is the neglect of time derivative $\partial_t \rho$ in the continuity equation that allows us to exclude from the consideration acoustic degrees of freedom and retain only "soft" modes as the internal waves which are conditioned by the relatively small difference of the two equilibrium density profiles. Compressibility of the medium in this model is manifested in form that a volume of each fluid element at slow motion is effectively "adapted" to the equilibrium density $\bar{\rho}(z)$, expanding when going up and compressing when going down [since $\bar{\rho}'(z) < 0$].

Let the shape of disturbed interface be given by equation $z = \eta(\mathbf{x}, t)$, where $\mathbf{x} = (x, y)$ is the radius-vector in the horizontal plane, and let the boundary values of the velocity potentials be

$$\psi_{1,2}(\mathbf{x}, t) = \varphi_{1,2}(\mathbf{x}, \eta(x, y, t), t).$$

At the free interface, the normal component $V_n$ of the velocity field should be continuous, as well as the pressure. From these considerations, two kinematic conditions and one dynamic condition are derived, which determine evolution of the system:

$$\frac{\partial \varphi_1}{\partial n}\Big|_{z=\eta} = \frac{\partial \varphi_2}{\partial n}\Big|_{z=\eta} \equiv V_n, \tag{7}$$

$$\eta_t = V_n \sqrt{1 + (\nabla \eta)^2}, \tag{8}$$

$$\{\bar{\rho}[\varphi_{1,t} - \varphi_{2,t}] + \frac{\bar{\rho}}{2}[(\nabla \varphi_1)^2 - (\nabla \varphi_2)^2]\}\Big|_{z=\eta} + g \int_h^\eta [\bar{\rho}_1(z) - \bar{\rho}_2(z)] dz = 0. \tag{9}$$

It follows form Eq.(7) that $\psi_1$ and $\psi_2$ are related to each other by a linear integral dependence. Therefore, if we fix the difference $\psi(\mathbf{x},t) \equiv \psi_1 - \psi_2$, then each potential will be fully determined. It is possible to prove that equations of motion for the two main functions $\eta(\mathbf{x},t)$ and $\psi(\mathbf{x},t)$ possess the Hamiltonian structure

$$\bar{\rho}(\eta)\eta_t = \delta\mathcal{H}/\delta\psi, \qquad -\bar{\rho}(\eta)\psi_t = \delta\mathcal{H}/\delta\eta, \tag{10}$$

with the corresponding Lagrangian $\mathcal{L} = \int \bar{\rho}(\eta)\eta_t\psi\, d^2\mathbf{x} - \mathcal{H}\{\eta,\psi\}$. The Hamiltonian functional $\mathcal{H}\{\eta,\psi\}$ is given by the following expression:

$$\begin{aligned}\mathcal{H} &= \int d^2\mathbf{x} \int_0^{\eta(\mathbf{x})} \bar{\rho}(z)\frac{(\nabla\varphi_1)^2}{2}dz + \int d^2\mathbf{x} \int_{\eta(\mathbf{x})}^{+\infty} \bar{\rho}(z)\frac{(\nabla\varphi_2)^2}{2}dz + g\int W(\eta)d^2\mathbf{x} \\ &= \frac{1}{2}\int \bar{\rho}(\eta)\psi V_n d^2\mathbf{x} + g\int W(\eta)d^2\mathbf{x},\end{aligned} \tag{11}$$

where

$$W'(\eta) = \int_h^{\eta} [\bar{\rho}_1(z) - \bar{\rho}_2(z)]dz, \tag{12}$$

that is $\mathcal{H}$ is the sum of the kinetic energy and an effective potential energy.

## General form of the dispersion relation for linear waves

If particular solutions of Eq.(6) are known in the form of linear combinations

$$\varphi_{\mathbf{k}}(\mathbf{x}, z) = [A\Phi_k^{(-)}(z) + B\Phi_k^{(+)}(z)]e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{13}$$

with decaying at $z \to +\infty$ functions $\Phi_k^{(-)}(z)$, and with growing at $z \to +\infty$ functions $\Phi_k^{(+)}(z)$, then dispersion relation for low-amplitude internal waves is

$$\omega_k^2 = \tilde{g}(h)\frac{D_1(h,k)D_2(h,k)}{[D_2(h,k) + D_1(h,k)]}, \tag{14}$$

where $\tilde{g}(h)$ is a renormalized gravity acceleration: $\tilde{g}(h) = g[\bar{\rho}_1(h) - \bar{\rho}_2(h)]/\bar{\rho}(h)$, and

$$D_1(h,k) \ = \ \frac{\Phi_k'^{(+)}(h)\Phi_k'^{(-)}(0) - \Phi_k'^{(-)}(h)\Phi_k'^{(+)}(0)}{\Phi_k^{(+)}(h)\Phi_k'^{(-)}(0) - \Phi_k^{(-)}(h)\Phi_k'^{(+)}(0)}, \tag{15}$$

$$D_2(h,k) \ = \ -\frac{\Phi_k'^{(-)}(h)}{\Phi_k^{(-)}(h)}. \tag{16}$$

For example, in the case $p \approx C_1\rho^\gamma$ we have $\bar{\rho}(z) \approx C_2(z_0 - z)^{1/(\gamma-1)}$. Functions $\Phi_k^{(\pm)}(z)$ are expressed through the modified Bessel functions $I_\nu$ and $K_\nu$, with the index $\nu = [(\gamma - 1)^{-1} - 1]/2$:

$$\Phi_k^{(-)}(z) = [k(z_0 - z)]^{-\nu} I_\nu(k(z_0 - z)), \tag{17}$$

$$\Phi_k^{(+)}(z) = [k(z_0 - z)]^{-\nu} K_\nu(k(z_0 - z)). \tag{18}$$

## The Hamiltonian in terms of Green's function

In some cases another way can be suitable how to calculate the Hamiltonian. Since the kinetic energy takes the form $\mathcal{K} = \frac{1}{2}\int(\mathbf{j}\cdot\mathbf{v})d^2\mathbf{x}dz$, where $\mathbf{j} = \bar{\rho}\mathbf{v}$ is the divergence-free field of the current density, we can introduce for $\mathbf{j}$ a vector potential $\mathbf{A}$ which satisfies the equation

$$\text{curl}\,\frac{1}{\bar{\rho}(z)}\,\text{curl}\,\mathbf{A} = \mathbf{\Omega} \equiv \text{curl}\,\mathbf{v}, \tag{19}$$

with the boundary condition $[\partial_x A^{(y)}(x,y,0) - \partial_y A^{(x)}(x,y,0)] = 0$. After that the kinetic energy can be re-written as follows,

$$\mathcal{K} = \frac{1}{2}\int \mathbf{A}\cdot\mathbf{\Omega}\ d^2\mathbf{x}dz = \frac{1}{2}\int G_{ik}(\mathbf{r}_1,\mathbf{r}_2)\Omega_i(\mathbf{r}_1)\Omega_k(\mathbf{r}_2)\ d^3\mathbf{r}_1 d^3\mathbf{r}_2, \tag{20}$$

where $G_{ik}(\mathbf{r}_1,\mathbf{r}_2)$ is the Green's function for Eq.(19). As far as the (singular) vorticity field $\mathbf{\Omega}$ is totally concentrated at the interface $z = \eta(\mathbf{x})$, and the vortex lines coincide with levels of the function $\psi(\mathbf{x})$ at that surface, the half-space integration will reduce to integration along the surface $z = \eta(\mathbf{x})$ by means of the change

$$(\Omega^{(x)},\Omega^{(y)},\Omega^{(z)})d^3\mathbf{r} \to (\psi_y, -\psi_x, \psi_y\eta_x - \psi_x\eta_y)dxdy. \tag{21}$$

As the simplest example, in this work an exponential profile

$$\bar{\rho}(z) = \rho_0 \exp(-2\alpha z)$$

of the equilibrium density is considered, when Eq.(19) after substitution $\mathbf{A} = \rho_0 e^{-2\alpha z}\mathbf{F}$ turns into an equation with constant coefficients. Generally speaking, if taken globally, such a dependence contradicts to adiabatic equations of state for real gases. Nevertheless, locally on the vertical coordinate near $z = h$, every realistic dependence $\bar{\rho}(z)$ is approximated by an exponent, provided not very long waves are considered.

Accordingly, the kinetic energy of the 3D system, without taking into account the flat rigid boundary, is given by the following expression in terms of $\eta$ and $\psi$,

$$\mathcal{K} = \frac{\rho_0}{8\pi}\int \frac{\exp[-\alpha\sqrt{|\mathbf{x}_1-\mathbf{x}_2|^2 + (\eta_1-\eta_2)^2} - \alpha(\eta_1+\eta_2)]}{\sqrt{|\mathbf{x}_1 - \mathbf{x}_2|^2 + (\eta_1 - \eta_2)^2}} \times\{\nabla\psi_1 \cdot \nabla\psi_2 + [\nabla\psi_1 \times \nabla\eta_1] \cdot [\nabla\psi_2 \times \nabla\eta_2]\} d^2\mathbf{x}_1 d^2\mathbf{x}_2, \qquad (22)$$

where $\nabla\eta$ and $\nabla\psi$ are 2D gradients. More cumbersome expression in the presence of the boundary $z = 0$ is also known.

## 2D potential flows

The expression for the kinetic energy of the two-layer 2D flow looks as follows:

$$\mathcal{K}_{2D} = \frac{\rho_0}{4\pi}\int [K_0\left(\alpha\sqrt{(x_1-x_2)^2+(\eta_1-\eta_2)^2}\right) - K_0\left(\alpha\sqrt{(x_1-x_2)^2+(\eta_1+\eta_2)^2}\right)]e^{-\alpha(\eta_1+\eta_2)}\psi_1'\psi_2' dx_1 dx_2, \quad (23)$$

where $\psi' = \partial\psi/\partial x$. We can also represent this functional in a slightly different form:

$$\mathcal{K}_{2D} = \frac{\rho_0}{2}\int dx_1 dx_2 \psi_1'\psi_2' e^{-\alpha(\eta_1+\eta_2)} e^{ik(x_1-x_2)} \times \int \frac{[e^{-|\eta_1-\eta_2|\sqrt{k^2+\alpha^2}} - e^{-(\eta_1+\eta_2)\sqrt{k^2+\alpha^2}}]}{2\sqrt{k^2+\alpha^2}}\frac{dk}{2\pi}. \quad (24)$$

From here we easily derive the dispersion relation,

$$\omega_k^2 = \tilde{g}(h)k^2\frac{[1-e^{-2h\sqrt{k^2+\alpha^2}}]}{2\sqrt{k^2+\alpha^2}}. \quad (25)$$

Now we consider the limiting case $\alpha\eta \ll 1$ and typical wave numbers $k$ satisfying the conditions $\alpha\eta \lesssim k\eta \ll 1$. Expanding the exponents in integral (24) in powers of the small arguments, we obtain an approximate kinetic energy functional up to the first order in $\alpha\eta$,

$$\mathcal{K}_*\{\eta, \psi\} = \frac{\rho_0}{2}\int \eta(1 - 2\alpha\eta)(\psi')^2 dx - \frac{\rho_0}{2}\int (\psi'\eta)[-\hat{\partial}_x^2 + \alpha^2]^{1/2}(\psi'\eta)\, dx. \quad (26)$$

Considering propagation of relatively small but finite disturbances $\tilde{\eta}(x,t) = \eta(x,t) - h$, it is possible by a standard procedure to derive weakly nonlinear equation for $u(x,t) = \psi_x$, which describes a slow evolution of uni-directional wave under the influence of weak dispersion:

$$u_t + \bar{c}u_x + \bar{a}uu_x - \frac{\bar{c}h}{2}\{[-\hat{\partial}_x^2 + \alpha^2]^{1/2} - \alpha\}u_x = 0, \quad (27)$$

where the speed of long linear waves is $\bar{c} \approx [\tilde{g}(0)h]^{1/2}$, and the coefficient $\bar{a} \approx 3/2$. It is interesting to note that the special form of the dispersive term makes the above equation intermediate between the two famous integrable models, namely the Korteweg-de Vries equation and the Benjamin-Ono equation.

## Planar flows with piecewise constant potential vorticity

Now we would like to make an important generalization of the Hamiltonian theory which is possible for 2D isentropic flows [in $(x, z)$ plane], namely we will take into account the fact that potential vorticity $\tilde{\gamma} = -\Omega^{(y)}/\rho$ in the 2D case is governed by the advection equation

$$\tilde{\gamma}_t + \mathbf{v} \cdot \nabla \tilde{\gamma} = 0. \tag{28}$$

Let (sufficiently small) constant potential vorticities in the layers be $\gamma_{1,2}$. We shall suppose that in the stationary state the velocity profile has a "break" at $z = h$, that is $U_{1,2}(z) = -\gamma_{1,2}\mu(z)$, where

$$\mu(z) = \int_h^z \bar{\rho}(\xi) d\xi. \tag{29}$$

A 2D velocity field in each layer now takes the form

$$\mathbf{v}_{1,2}(x, z, t) = (U_{1,2}(z) + \partial_x \varphi_{1,2}(x, z, t), \quad \partial_z \varphi_{1,2}(x, z, t)), \tag{30}$$

with the potentials $\varphi_{1,2}$ satisfying the same equation (6): $\nabla \cdot \bar{\rho} \nabla \varphi_{1,2} = 0$, and it implies the existence of the corresponding stream functions $\vartheta_{1,2}(x, z, t)$:

$$\bar{\rho} \partial_x \varphi_{1,2} = \partial_z \vartheta_{1,2}, \qquad \bar{\rho} \partial_z \varphi_{1,2} = -\partial_x \vartheta_{1,2}. \tag{31}$$

The full stream functions of the flows under consideration are

$$\Theta_{1,2}(x,z,t) = \vartheta_{1,2}(x,z,t) - U_{1,2}^2(z)/(2\gamma_{1,2}), \tag{32}$$

Instead of the approximate Bernoulli equation, we have to deal with its generalization:

$$\partial_t \varphi_{1,2} + \gamma_{1,2}\Theta_{1,2} + \frac{(\mathbf{v}_{1,2})^2}{2} + \frac{\tilde{p}_{1,2}}{\bar{\rho}(z)} = 0. \tag{33}$$

which regards the 2D Euler equation in the case of constant potential vorticity under the condition $\nabla \cdot (\bar{\rho}\mathbf{v}) = 0$.

At the interface $z = \eta(x,t)$ there are the equalities

$$-\partial_x \Theta_1(x,\eta(x)) = -\partial_x \Theta_2(x,\eta(x)) = \bar{\rho}(\eta)\eta_t = \bar{\rho}(\eta) V_n \sqrt{1+\eta'^2}, \tag{34}$$

where $V_n = (\mathbf{v}_1 \cdot \mathbf{n}) = (\mathbf{v}_2 \cdot \mathbf{n})$.

Demanding the pressure field to be continuous at $z = \eta(x,t)$ and reasoning analogously to the case $\gamma_{1,2} = 0$, we conclude that the evolution equations for the 2D system possess the following structure,

$$\bar{\rho}(\eta)\eta_t = \delta\mathcal{H}/\delta\psi, \tag{35}$$

$$-\bar{\rho}(\eta)\psi_t + \gamma\bar{\rho}(\eta)\partial_x^{-1}[\bar{\rho}(\eta)\eta_t] = \delta\mathcal{H}/\delta\eta, \tag{36}$$

where $\gamma = (\gamma_1 - \gamma_2)$, and the Hamiltonian $\mathcal{H}$ is equal to the sum of total kinetic energy and the effective potential energy. The corresponding Lagrangian for the above equations is

$$\mathcal{L} = \int \psi\mu_t\, dx + \frac{\gamma}{2}\int \mu\partial_x^{-1}\mu_t\, dx - \mathcal{H}\{\mu,\psi\}, \tag{37}$$

where $\mu = \mu(\eta) = \int_h^\eta \bar{\rho}(z)dz$.

In particular, it is easy to show that the dispersion relation for linear waves with $\gamma \neq 0$ is given by the following formula,

$$\omega_k = \gamma\bar{\rho}(h)kN(h,k)/2 + \sqrt{[\gamma\bar{\rho}(h)kN(h,k)]^2/4 + \tilde{g}(h)k^2N(h,k)}. \tag{38}$$

## Discussion

Promising directions of further research can be outlined as follows.

1) Generalization of the model is evident for more layers and for continuous limit. Interaction between several interfaces in many cases is able to introduce new interesting effects as instabilities etc.

2) We should mention a wide class of problems concerning interaction of internal waves with mountains, which also can be studied with the help of this model.

3) Nonlinear wave dynamics can be simulated numerically.

4) An analogous Hamiltonian formulation is possible for consideration of axisymmetric flows with a piecewise constant generalized potential vorticity.

5) It seems likely that analogous finite-layer models are possible not only in the Eulerian hydrodynamics, but in a wider class of conservative hydrodynamic systems as well, for instance, in the hydrodynamics of a relativistic fluid placed in a strong static gravitational field described by a metric 4-tensor. Accordingly, there is a perspective of application of a similar theory to astrophysical problems (where the equilibrium density possesses the spherical symmetry, as a rule).

**THANK YOU FOR ATTENTION!**