Topological matter
insulators, topololgical superfluid ^{3}He-B & relativistic quantum vacuum

Aalto-yliopisto

G. Volovik

Landau Institute

RUSSIAN ACADEMY OF SCIENCES
L.D Landau
INSTITUTE FOR
THEORETICAL
PHYSICS

Chernogolovka, June 21 2010

1. Introduction

* topological matter and topological quantum phase transitions (TQPT)

2. Fully gapped 2D topological media

* films of superfluid 3He-A and planar phase, 2D topological insulators

* topological invariants for gapped 2D topological matter

* edge states & fermion zero modes

3. Fully gapped 3D topological media

* superfluid 3He-B, topological insulators, vacuum of Standard Model of particle physics

* topological invariants for gapped 3D topological matter

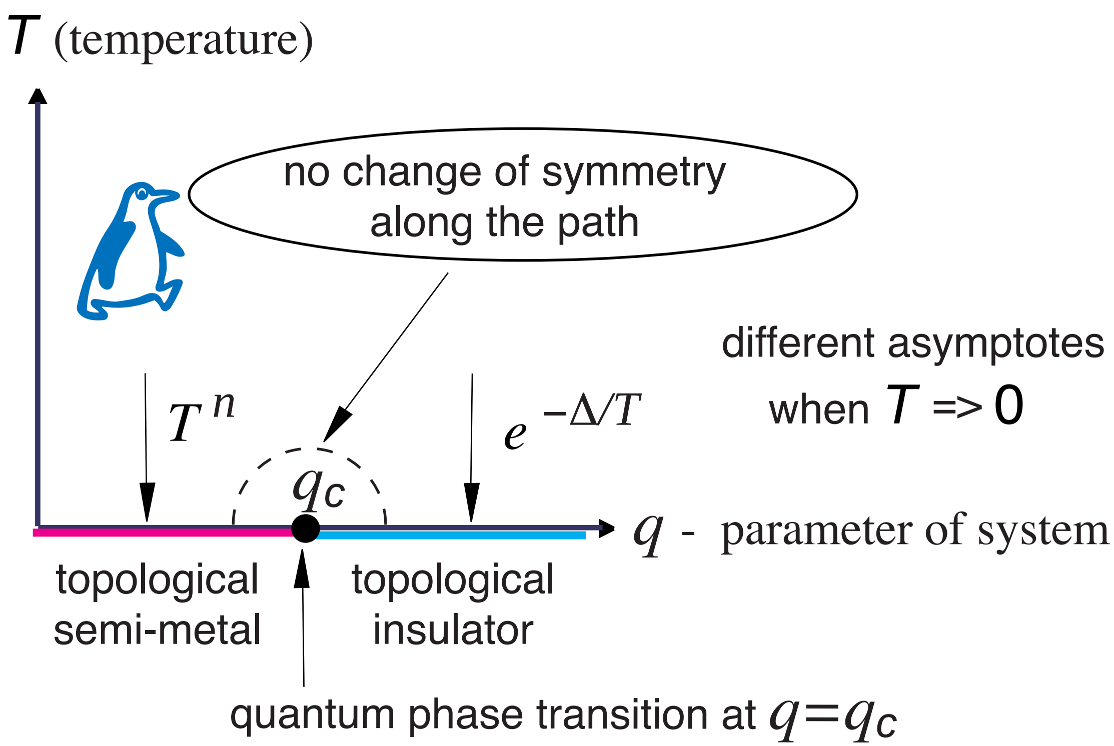
* edge states & Majorana fermions

4. Fermion zero modes on vortices

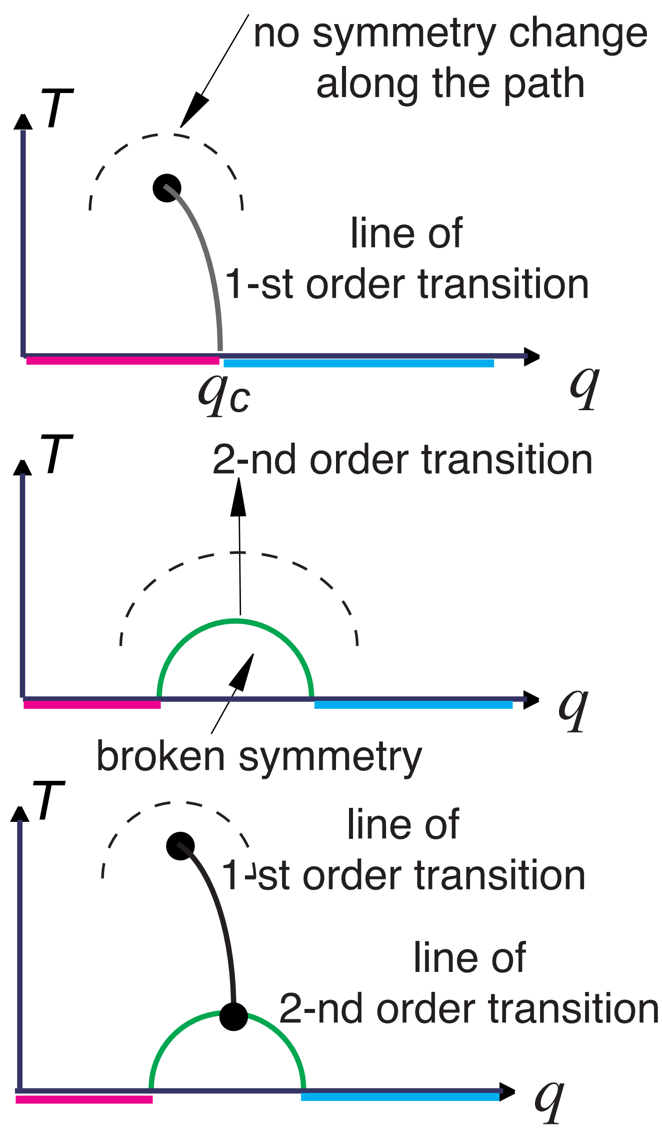
topological quantum phase transitions

transitions between **ground states (vacua)** of the **same symmetry**,
but **different topology** in **momentum space**

example: QPT between gapless & gapped matter



QPT interrupted by thermodynamic transitions



other topological QPT:
Lifshitz transition,
transtion between topological and nontopological superfluids,
plateau transitions,
confinement-deconfinement transition, ...

topological insulators & superconductors in 2+1

p-wave 2D superconductor, ^{3}He-A film, HgTe insulator quantum well

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$p^2 = p_x{}^2 + p_y{}^2$$

How to extract useful information on energy states from Hamiltonian without solving equation

$$H\psi = E\psi$$

Topological invariant in momentum space

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

$$H = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

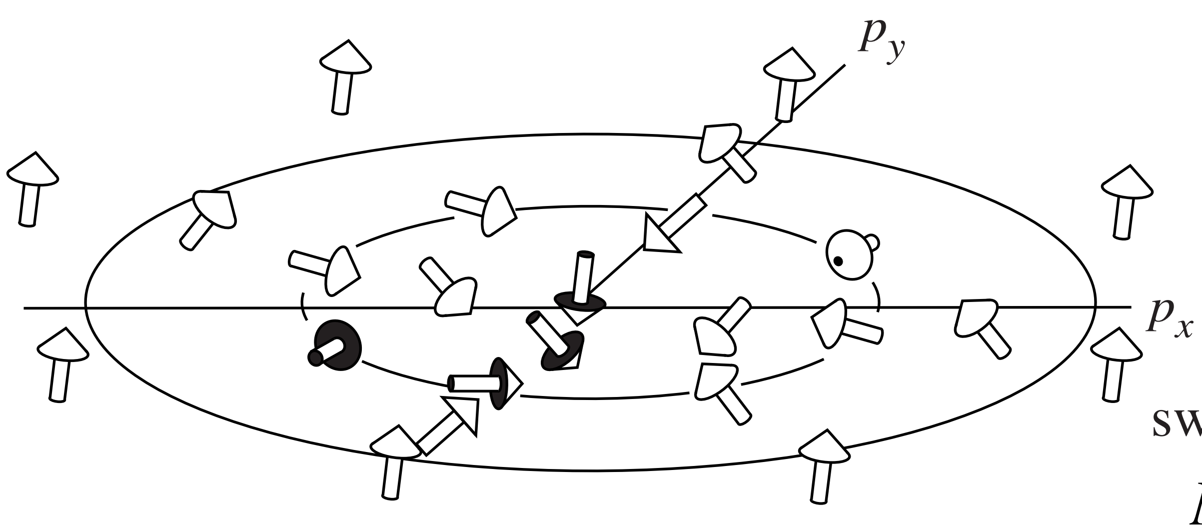
$$p^2 = p_x{}^2 + p_y{}^2$$

fully gapped 2D state at $\mu \neq 0$

$$\tilde{N}_3 = \frac{1}{4\pi} \int d^2p \; \hat{\mathbf{g}} \cdot (\boldsymbol{\partial}_{p_x} \hat{\mathbf{g}} \times \boldsymbol{\partial}_{p_y} \hat{\mathbf{g}})$$

GV, JETP **67**, 1804 (1988)

Skyrmion (coreless vortex) in momentum space at $\mu > 0$

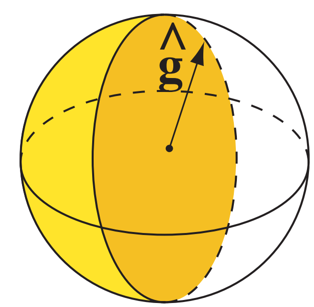


unit vector

$\hat{\mathbf{g}}(p_x, p_y)$

sweeps unit sphere

$\tilde{N}_3\ (\mu > 0) = 1$

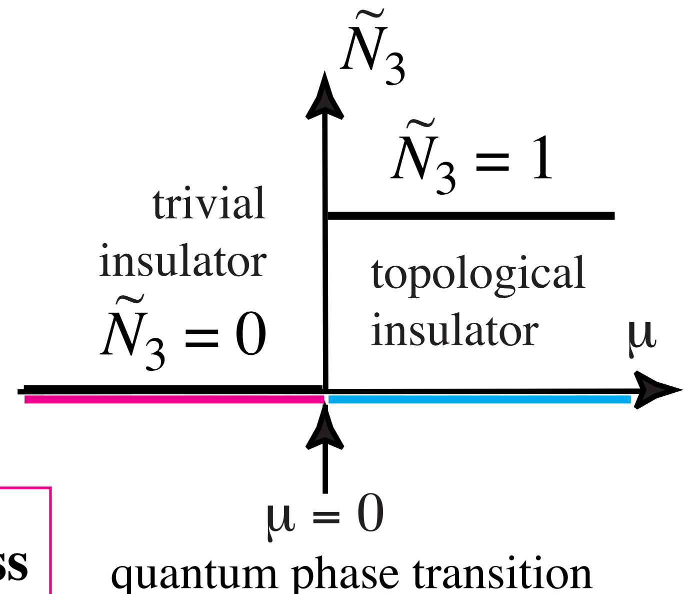


quantum phase transition:
from topological to non-topologicval insulator/superconductor

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + ip_y) \\ c(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} = \begin{pmatrix} g_3(\mathbf{p}) & g_1(\mathbf{p}) + i\, g_2(\mathbf{p}) \\ g_1(\mathbf{p}) - i\, g_2(\mathbf{p}) & -g_3(\mathbf{p}) \end{pmatrix} = \boldsymbol{\tau} \cdot \mathbf{g}(\mathbf{p})$$

Topological invariant in momentum space

$$\tilde{N}_3 = \frac{1}{4\pi} \int d^2p \ \hat{\mathbf{g}} \cdot (\boldsymbol{\partial}_{p_x} \hat{\mathbf{g}} \times \boldsymbol{\partial}_{p_y} \hat{\mathbf{g}})$$



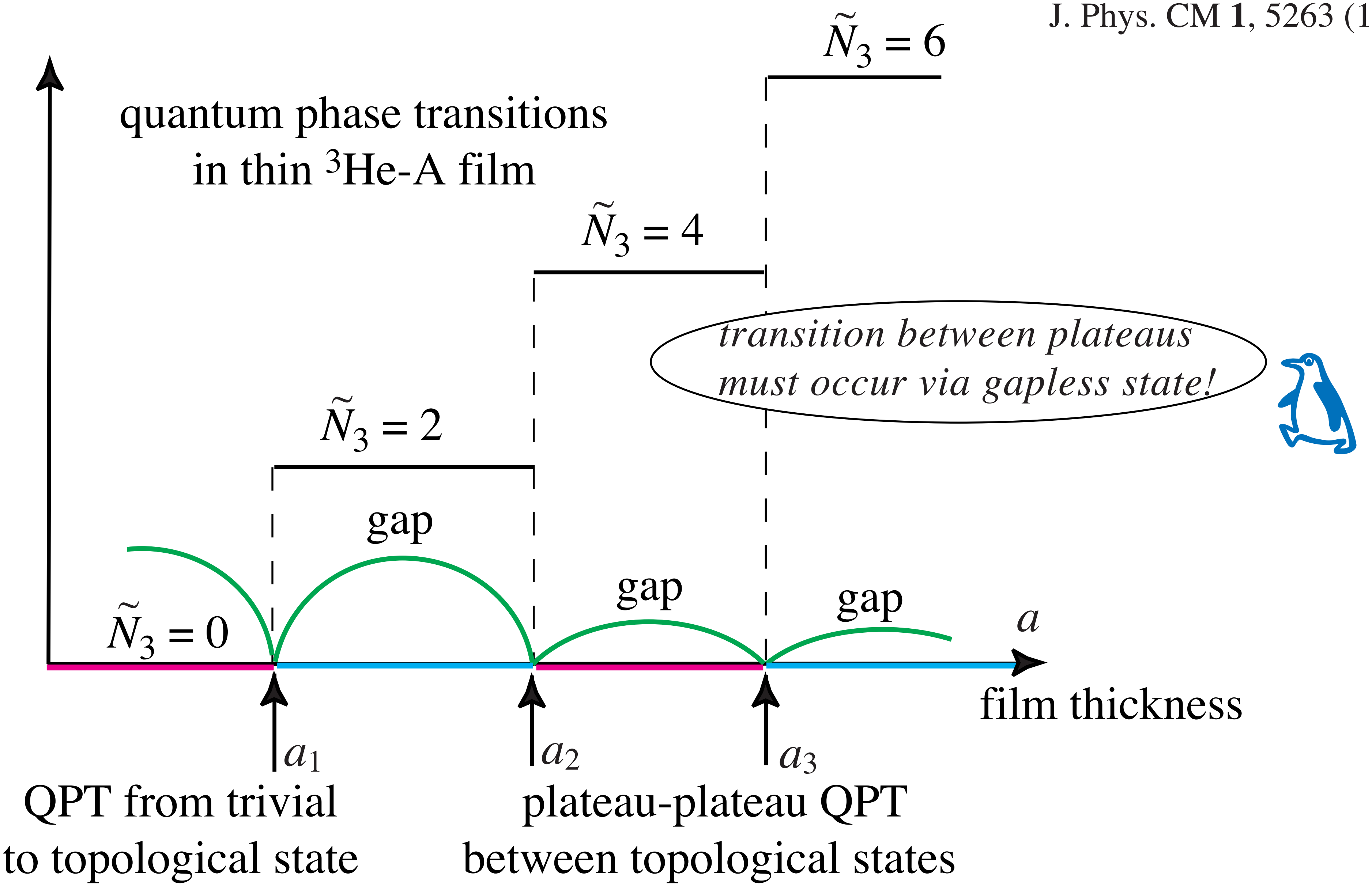
intermediate state at $\mu = 0$ must be gapless

$\Delta\tilde{N}_3 \neq 0$ is origin of fermion zero modes
at the interface between states with different $\tilde{N}_3$

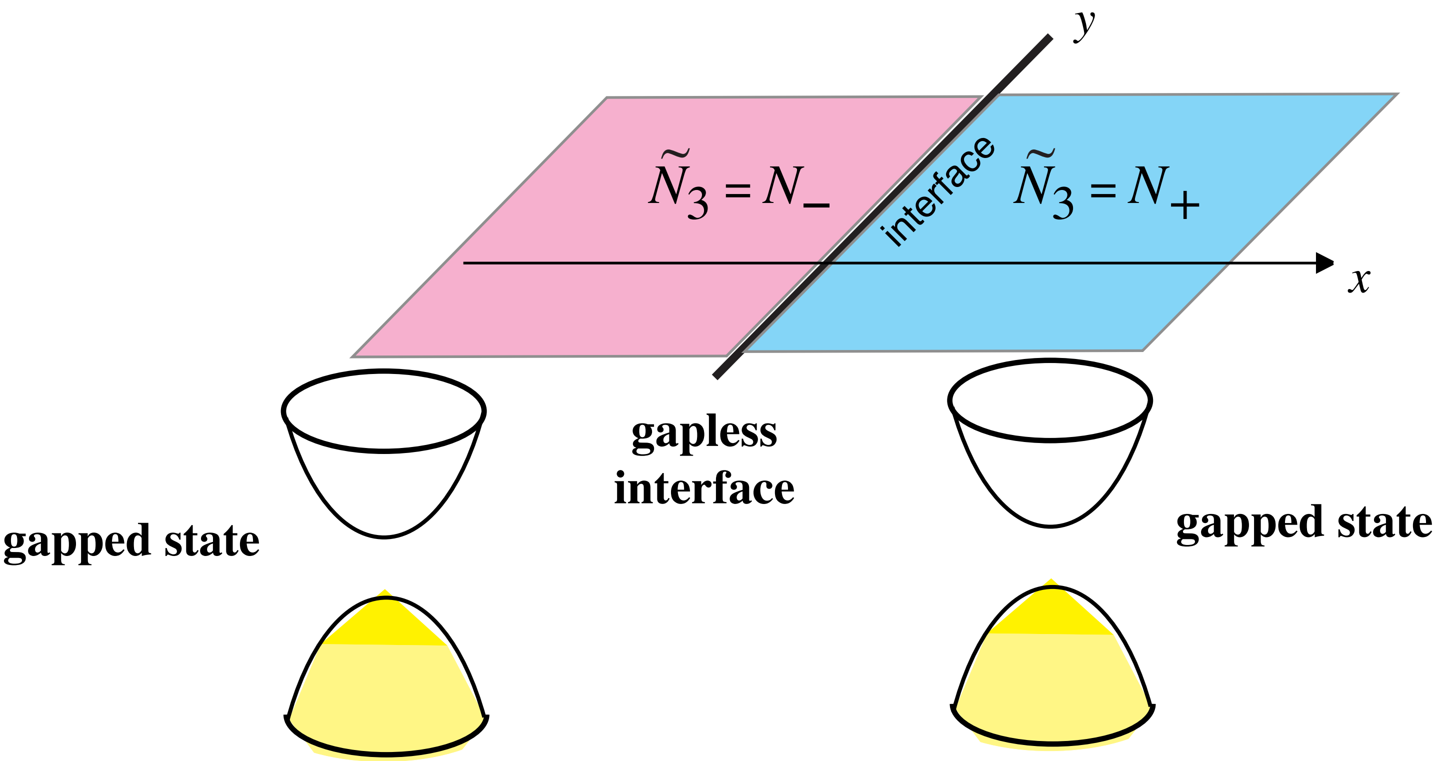
p-space invariant in terms of Green's function & topological QPT

$$\tilde{N}_3 = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \, \mathbf{tr} \int d^2p \, d\omega \, \mathbf{G} \, \partial^\mu \mathbf{G}^{-1} \, \mathbf{G} \, \partial^\nu \mathbf{G}^{-1} \mathbf{G} \, \partial^\lambda \mathbf{G}^{-1}$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

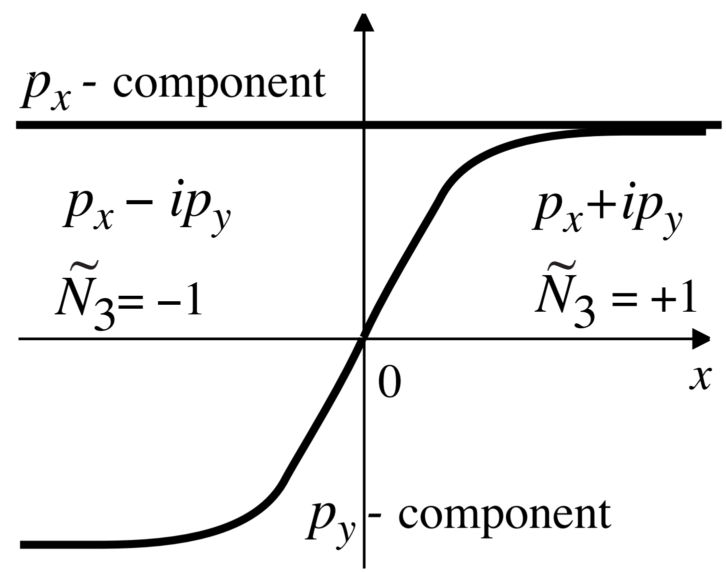


interface between two 2+1 topological insulators or gapped superfluids

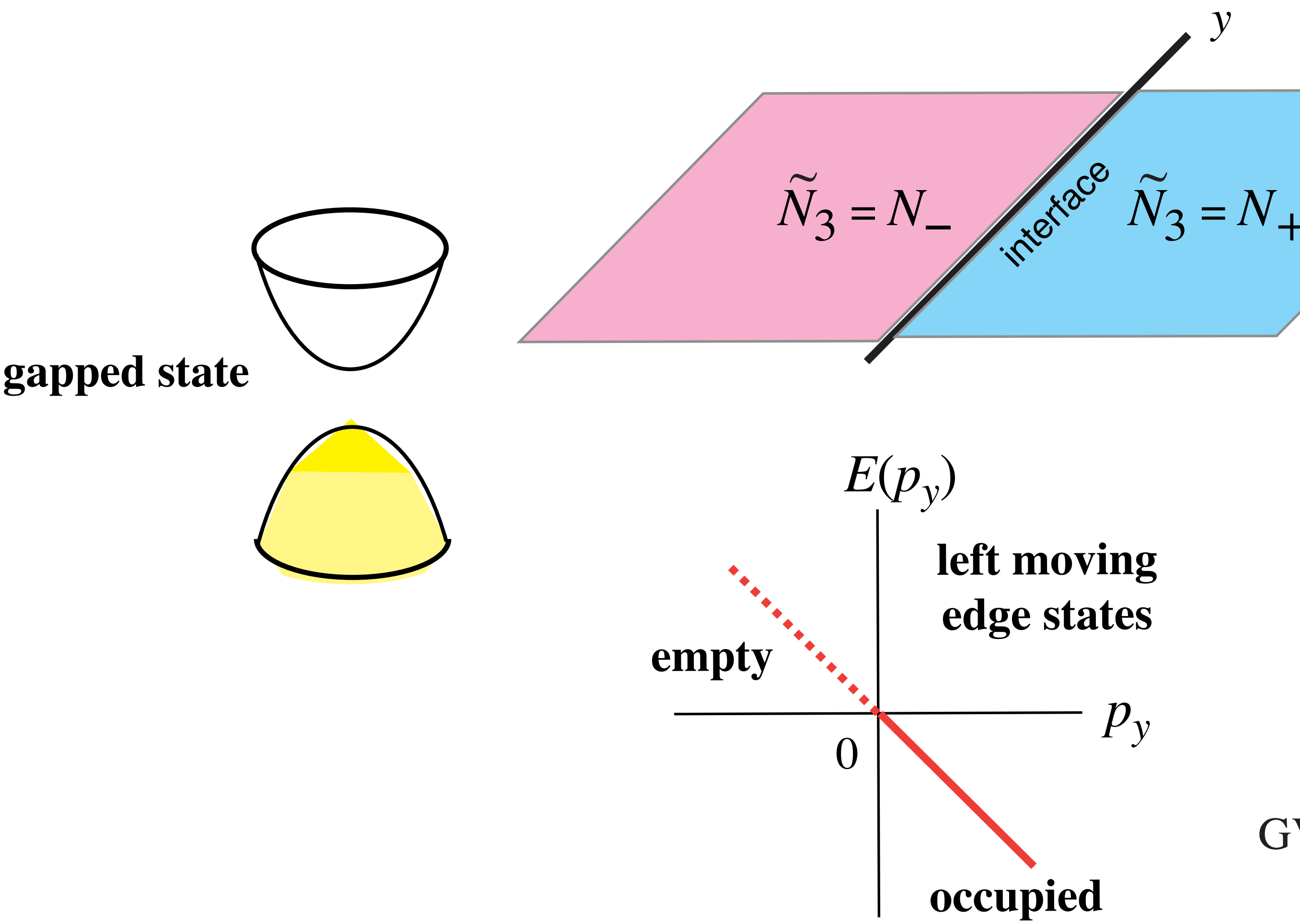


* domain wall in 2D chiral superconductors:

$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c(p_x + i\,p_y \tanh x) \\ c(p_x - i\,p_y \tanh x) & -\frac{p^2}{2m} + \mu \end{pmatrix}$$



Edge states at interface between two 2+1 topological insulators or gapped superfluids

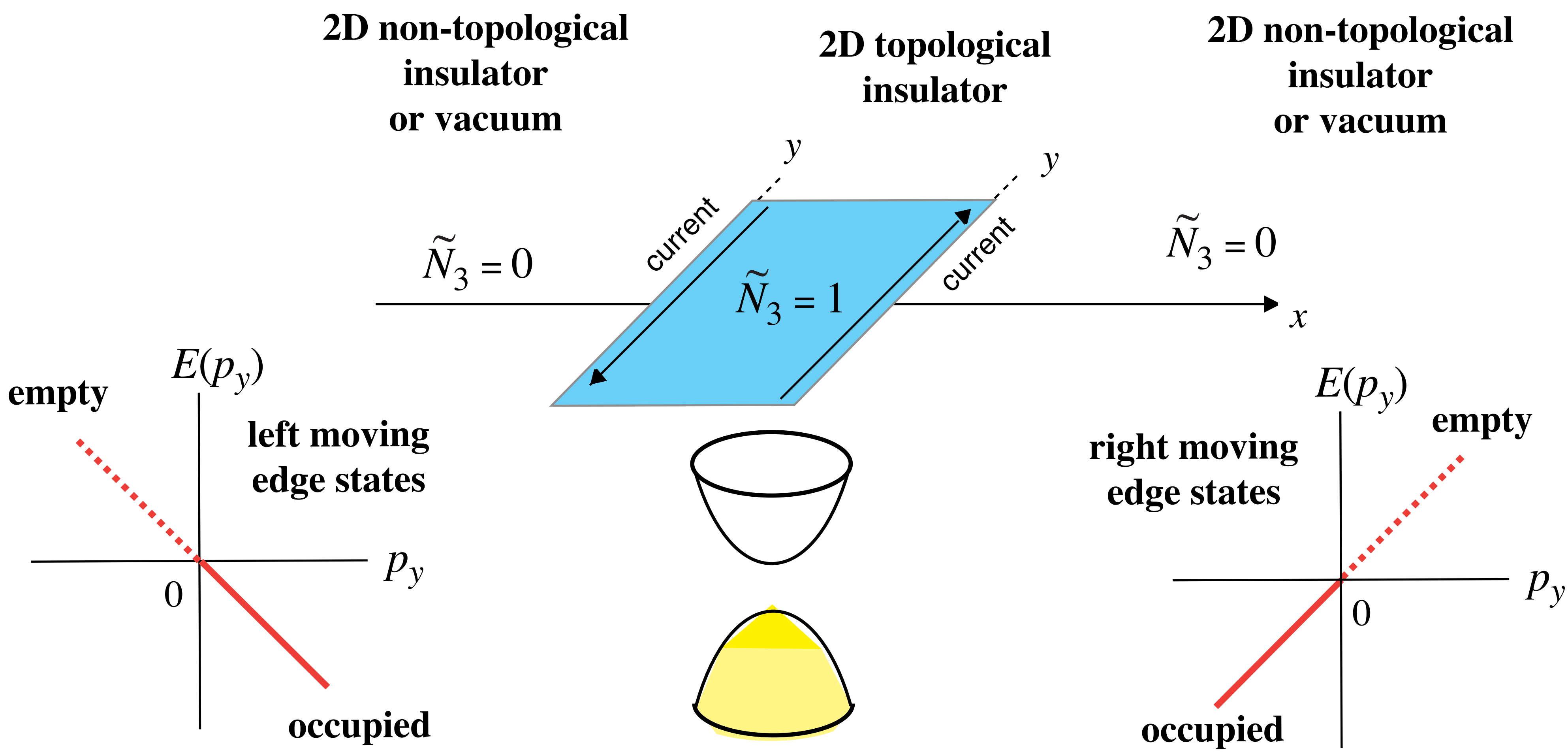


GV JETP Lett. **55**, 368 (1992)

Index theorem:
number of fermion zero modes
at interface:

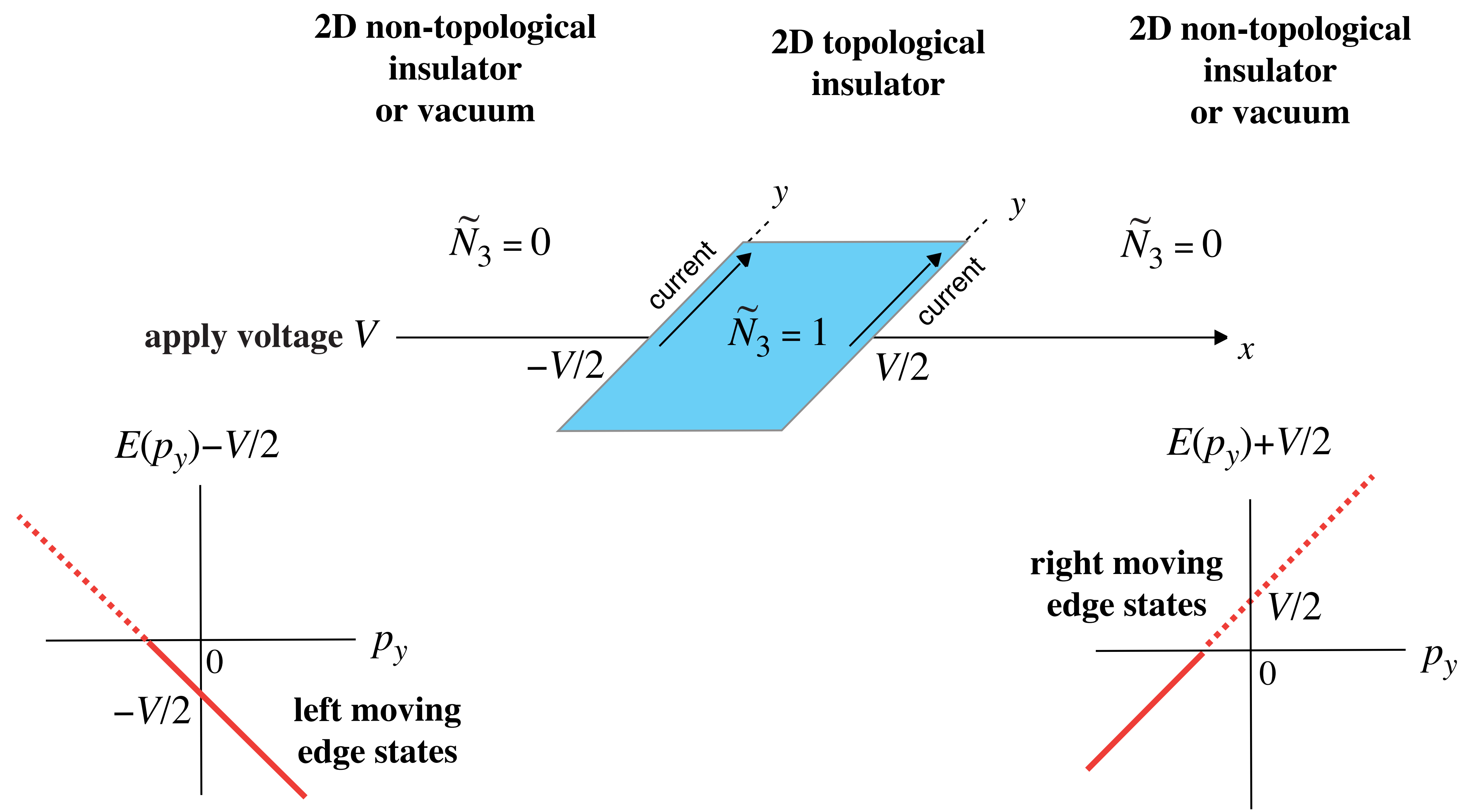
$$\nu = N_+ - N_-$$

Edge states and currents



current $J_y = J_{\text{left}} + J_{\text{right}} = 0$

Edge states and Quantum Hall effect



current $J_y = J_{\mathrm{left}} + J_{\mathrm{right}} = \sigma_{xy} E_y$

$$\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}_3$$

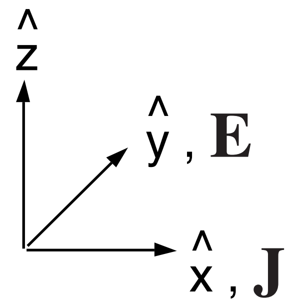
Intrinsic quantum Hall effect & momentum-space invariant

$$S_{CS} = \frac{e^2}{16\pi} \tilde{N}_3 e^{\mu\nu\lambda} \int d^2x \, dt \, A_\mu F_{\nu\lambda}$$

p-space invariant — **r**-space invariant

A_μ - electromagnetic field

electric current $J_x = \delta S_{CS} / \delta A_x = \dfrac{e^2}{4\pi} \tilde{N}_3 E_y$

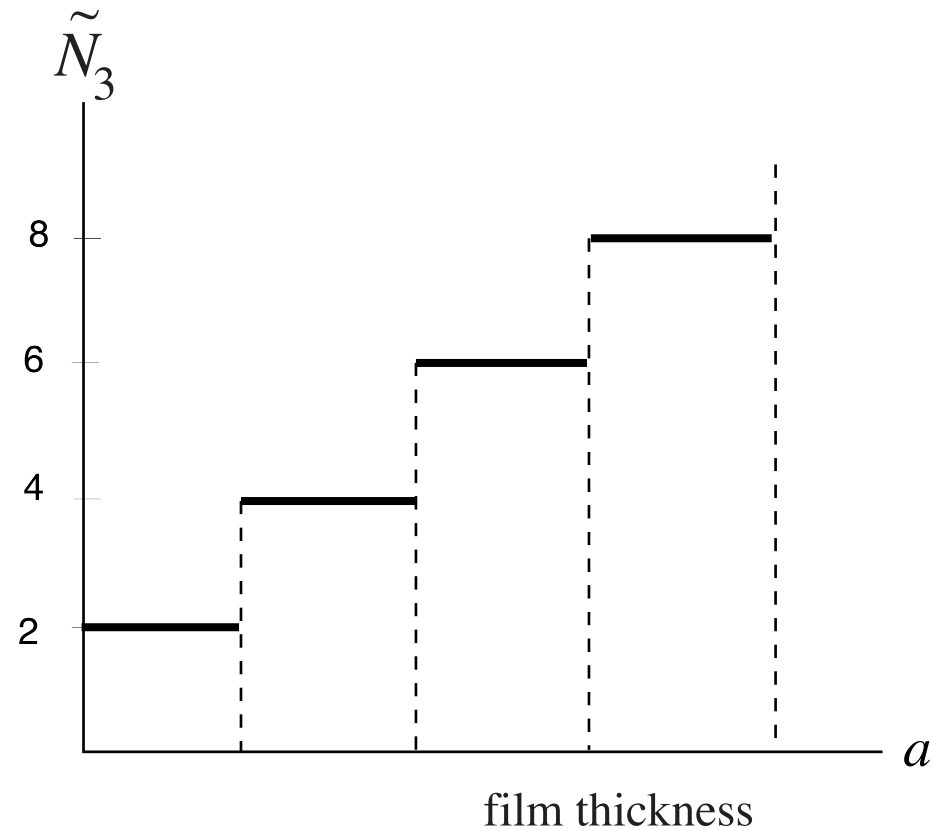


quantized intrinsic Hall conductivity
(without external magnetic field)

$$\sigma_{xy} = \frac{e^2}{4\pi} \tilde{N}_3$$

GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

film of topological quantum liquid



general Chern-Simons terms & momentum-space invariant

(interplay of ***r***-space and ***p***-space topologies)

$$S_{CS} = \frac{1}{16\pi} N_{IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ I} F_{\nu\lambda}^{\ \ J}$$

***r*-space invariant**

↓

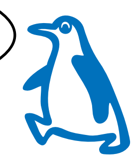
***p*-space invariant protected by symmetry**

↓

$$N_{IJ} = \frac{1}{24\pi^2} e_{\mu\nu\lambda} \mathbf{tr}\left[\int d^2p\, d\omega\, K_I K_J\, \mathbf{G}\, \boldsymbol{\partial}^\mu\, \mathbf{G}^{-1}\, \mathbf{G}\, \boldsymbol{\partial}^\nu\, \mathbf{G}^{-1} \mathbf{G}\, \boldsymbol{\partial}^\lambda\, \mathbf{G}^{-1}\right]$$

K_I - charge interacting with gauge field $A_\mu^{\ I}$

gauge fields can be real, artificial or auxiliary



$K=e$ for electromagnetic field A_μ

$K=\hat{\sigma}_z$ for effective spin-rotation field $A_\mu^{\ z}$ $\quad (A_0^{\ z}= \gamma H^{\ z})$

$$id/dt - \gamma\hat{\sigma}\cdot\mathbf{H} = id/dt - \hat{\sigma}\cdot\mathbf{A}_0$$

applied Pauli magnetic field plays the role of components of effective SU(2) gauge field $A_\mu^{\ i}$

Intrinsic spin-current quantum Hall effect & momentum-space invariant

$$S_{CS} = \frac{1}{16\pi} N_{IJ}\, e^{\mu\nu\lambda} \int d^2x\, dt\, A_\mu^{\ I} F_{\nu\lambda}^{\ J}$$

spin current $J_x^{\ z} = \delta S_{CS} / \delta A_x^{\ z} = \frac{1}{4\pi} (\gamma N_{ss}\, dH^z/dy + N_{se} E_y)$

spin-spin QHE (for the N_{ss} term)

spin-charge QHE (for the N_{se} term)

2D singlet superconductor:

$$\sigma_{xy}^{spin/spin} = \frac{N_{ss}}{4\pi}$$

s-wave: $N_{ss} = 0$

$p_x + ip_y$: $N_{ss} = 2$

$d_{xx-yy} + id_{xy}$: $N_{ss} = 4$

film of planar phase of superfluid ^{3}He

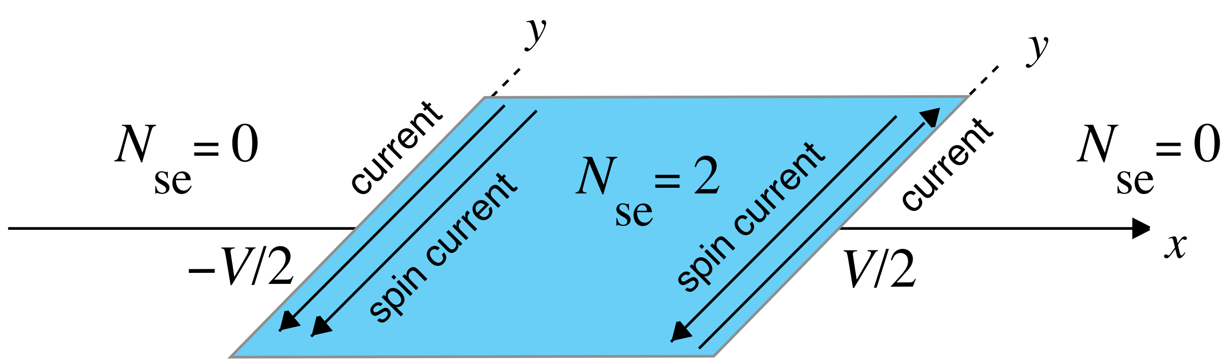
$$\sigma_{xy}^{spin/charge} = \frac{N_{se}}{4\pi}$$

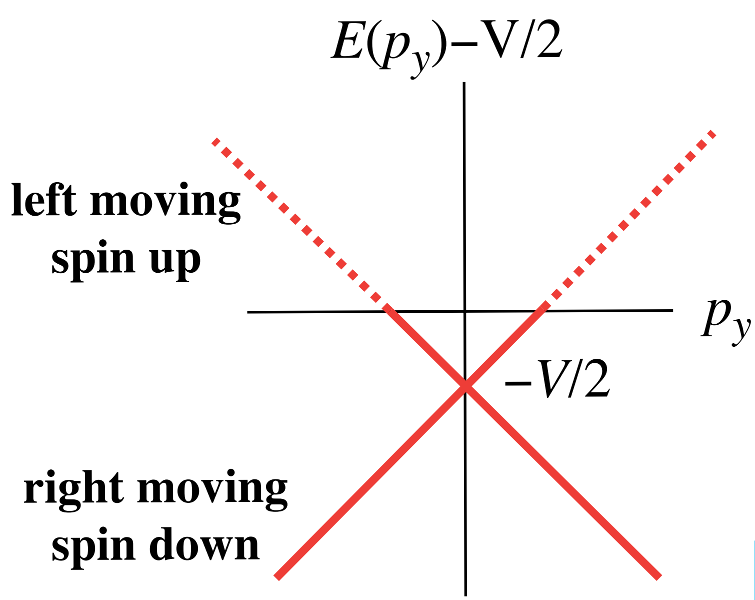
GV & Yakovenko
J. Phys. CM **1**, 5263 (1989)

Intrinsic spin-current quantum Hall effect & edge state

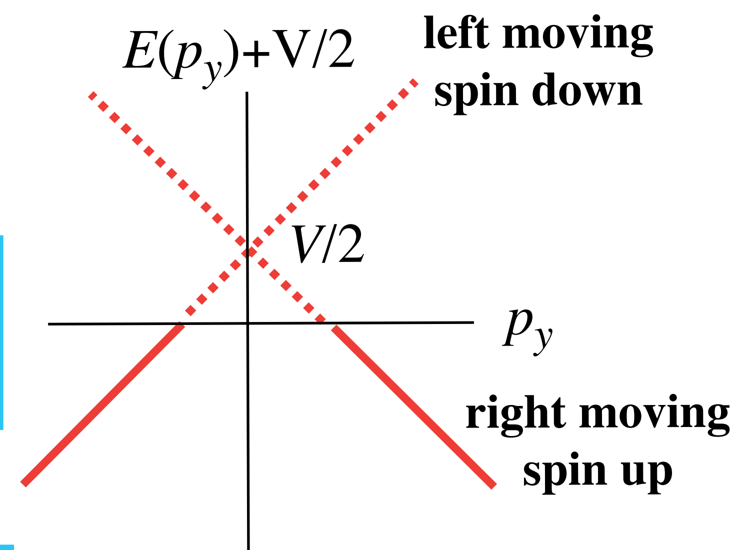
spin current $$J_x^{\,z} = \frac{1}{4\pi}\left(\gamma N_{ss}\, dH^z/dy + N_{se}\, E_y\right)$$

spin-charge QHE





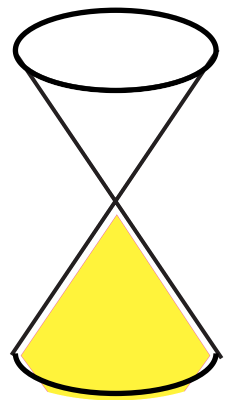
$$\sigma_{xy}^{\text{spin/charge}} = \frac{N_{se}}{4\pi}$$



electric current is zero
spin current is nonzero

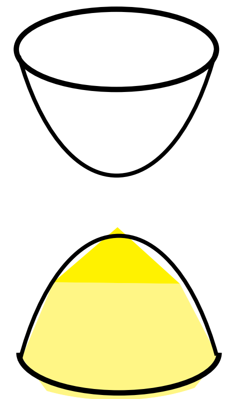
3D topological superfluids/insulators/semiconductors

gapless topologically
nontrivial vacua



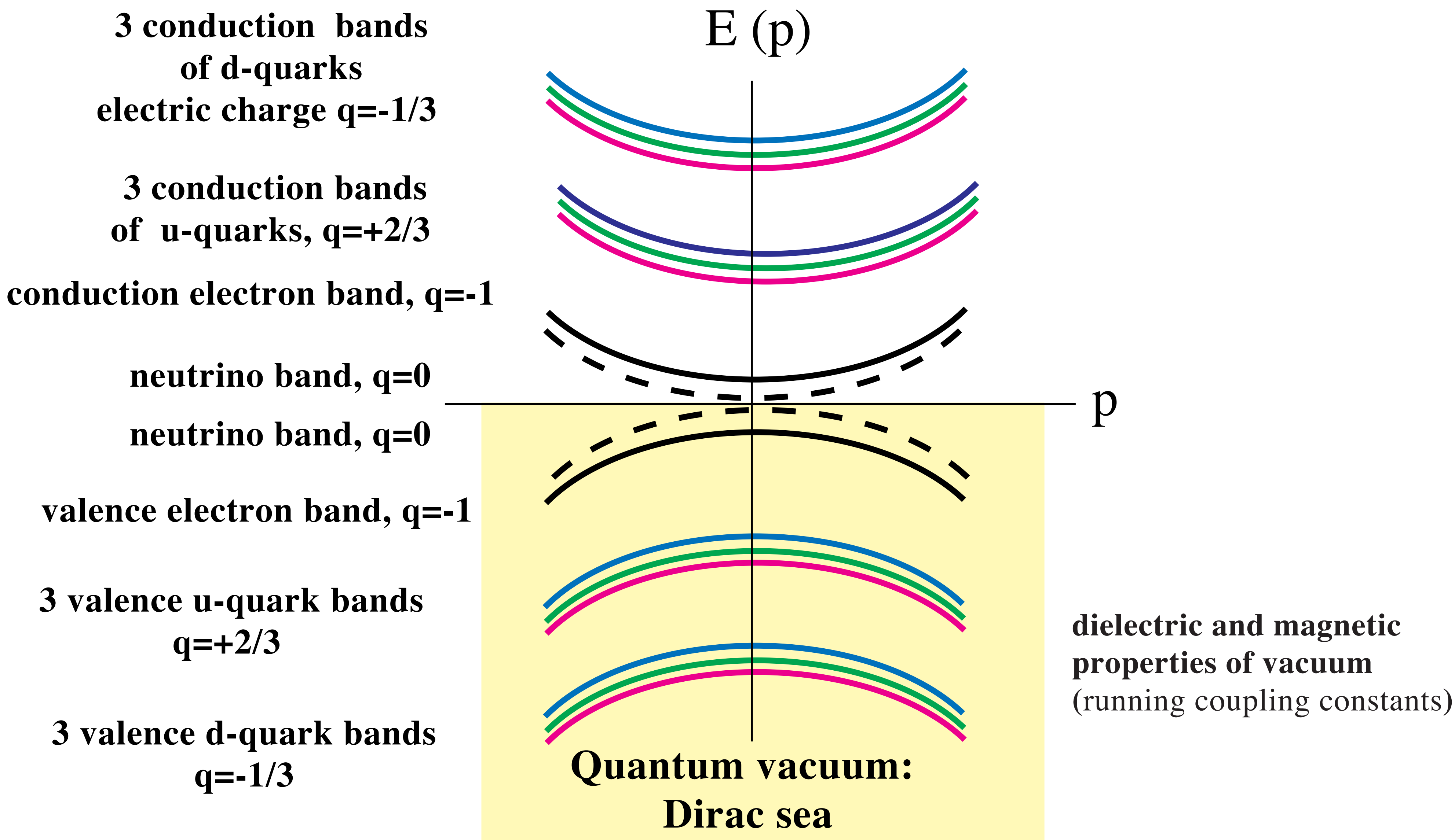
3He-A,
Standard Model above electroweak transition,
semimetals

fully gapped topologically
nontrivial vacua



3He-B,
Standard Model below electroweak transition,
topological insulators (Be_2Se_3 , ...),
triplet & singlet color/chiral superconductors, ...

Present vacuum as semiconductor or insulator



electric charge of quantum vacuum

$$Q = \sum_a q_a = N\,[-1 + 3\times(-1/3) + 3\times(+2/3)\,] = 0$$

fully gapped 3+1 topological matter

superfluid ^{3}He-B, topological insulator Bi_2Te_3 , present vacuum of Standard Model

*** Standard Model vacuum as topological insulator**

Topological invariant protected by symmetry

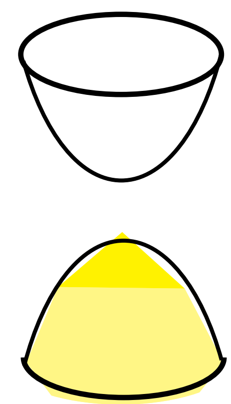
$$N_K = \frac{1}{24\pi^2} \, e_{\mu\nu\lambda} \, \mathbf{tr} \int dV \, K \, \mathbf{G} \, \boldsymbol{\partial}^\mu \, \mathbf{G}^{-1} \, \mathbf{G} \, \boldsymbol{\partial}^\nu \, \mathbf{G}^{-1} \mathbf{G} \, \boldsymbol{\partial}^\lambda \, \mathbf{G}^{-1}$$

over 3D momentum space

G is Green's function at $\omega=0$, K is symmetry operator **GK** =+/– **KG**

Standard Model vacuum: $K=\gamma_5$ $\mathbf{G}\gamma_5 = -\gamma_5\mathbf{G}$

$$N_K = 8n_g$$



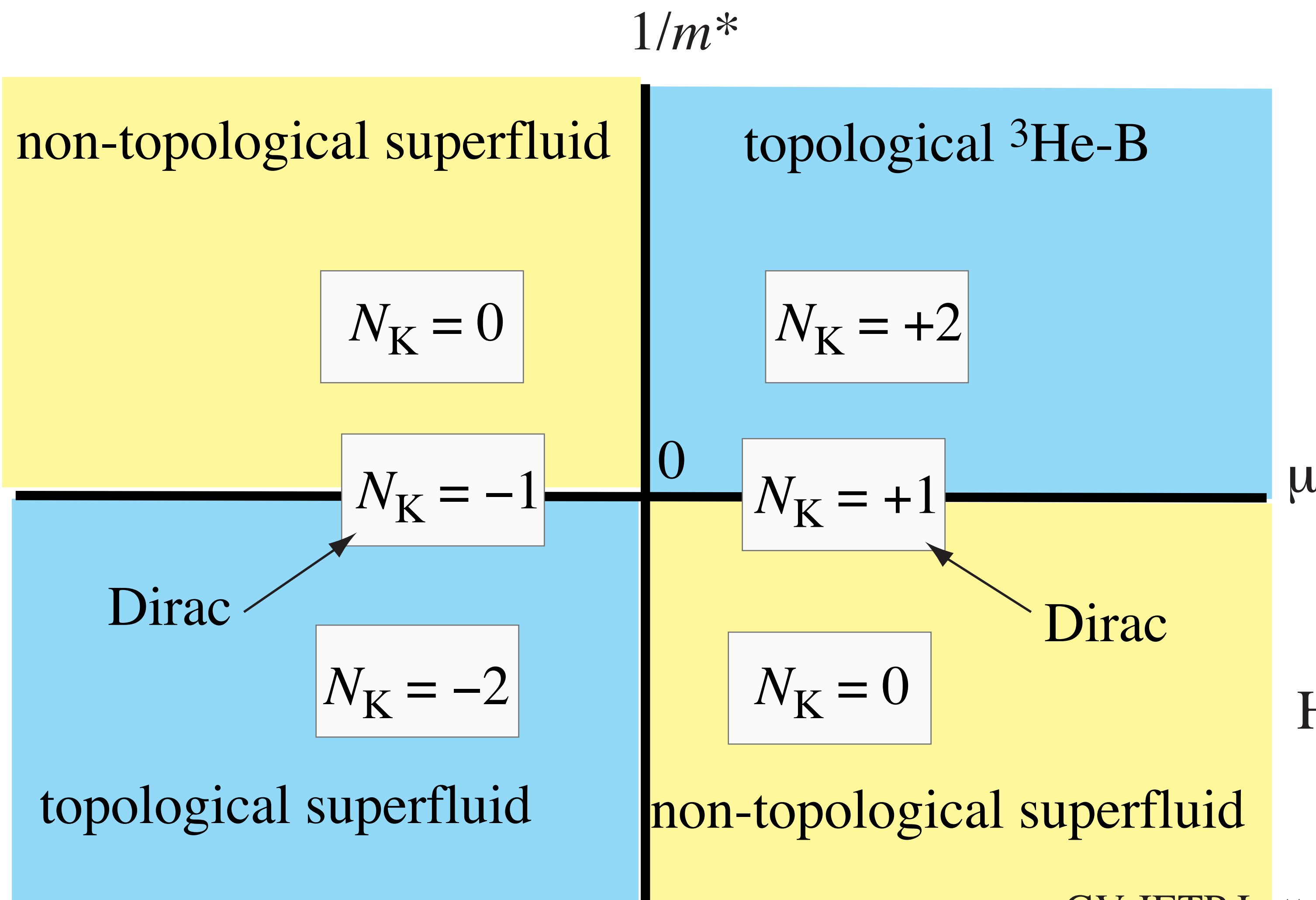
8 massive Dirac particles in one generation

topological superfluid ^{3}He-B

$$H\tau_2 = -\tau_2 H$$

$$K = \tau_2$$

$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix} = \left(\frac{p^2}{2m^*} - \mu\right)\tau_3 + c_B \sigma\cdot\mathbf{p}\,\tau_1$$



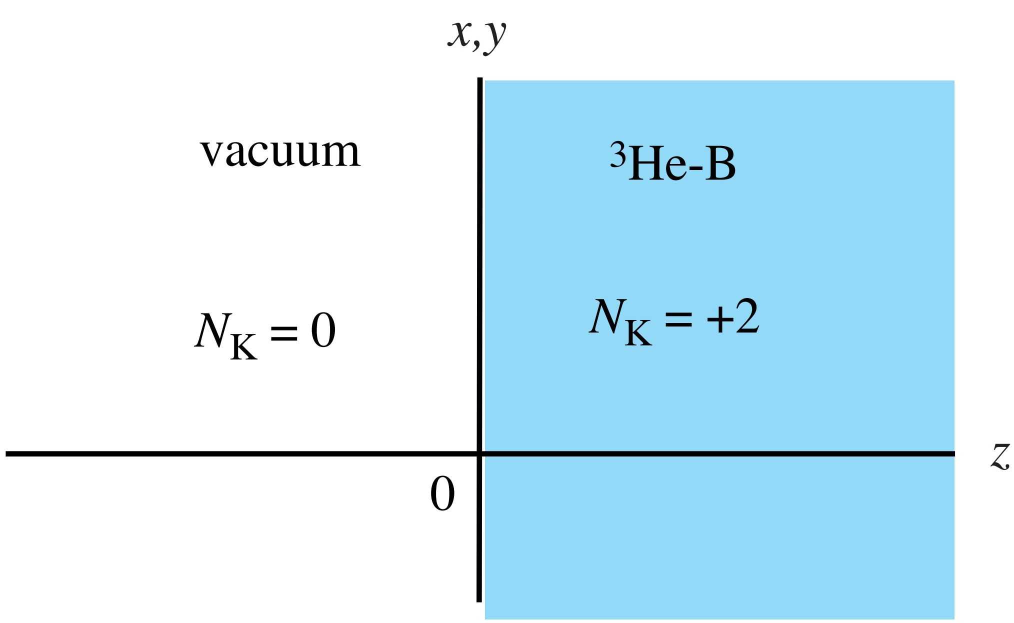
Dirac vacuum

$$1/m^* = 0$$

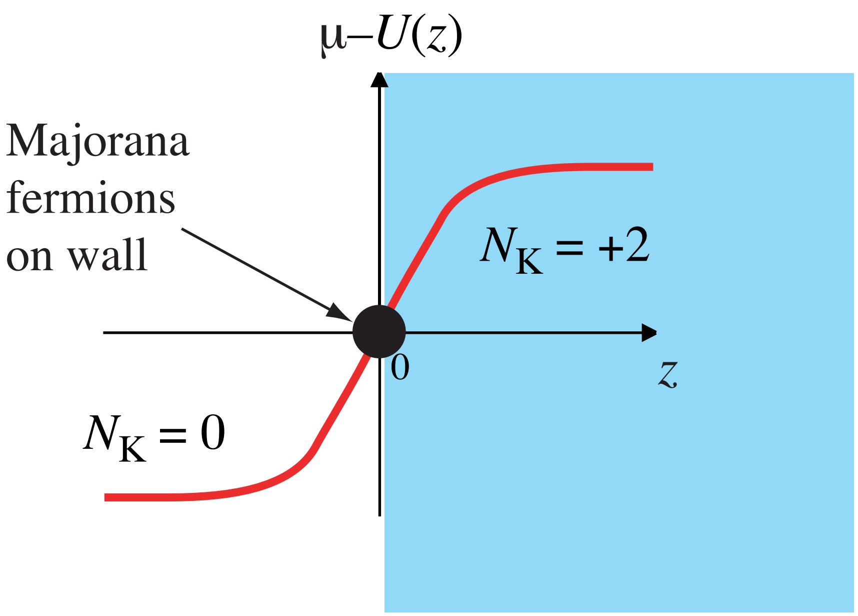
$$H = \begin{pmatrix} -M & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & +M \end{pmatrix}$$

GV JETP Lett. **90**, 587 (2009)

Boundary of 3D gapped topological superfluid



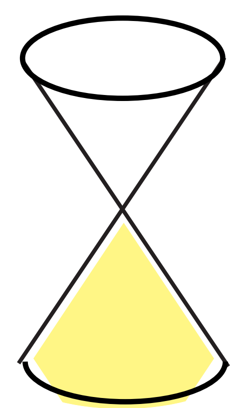
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu + U(z) & c_B \sigma \cdot \mathbf{p} \\ c_B \sigma \cdot \mathbf{p} & -\frac{p^2}{2m^*} + \mu - U(z) \end{pmatrix}$$



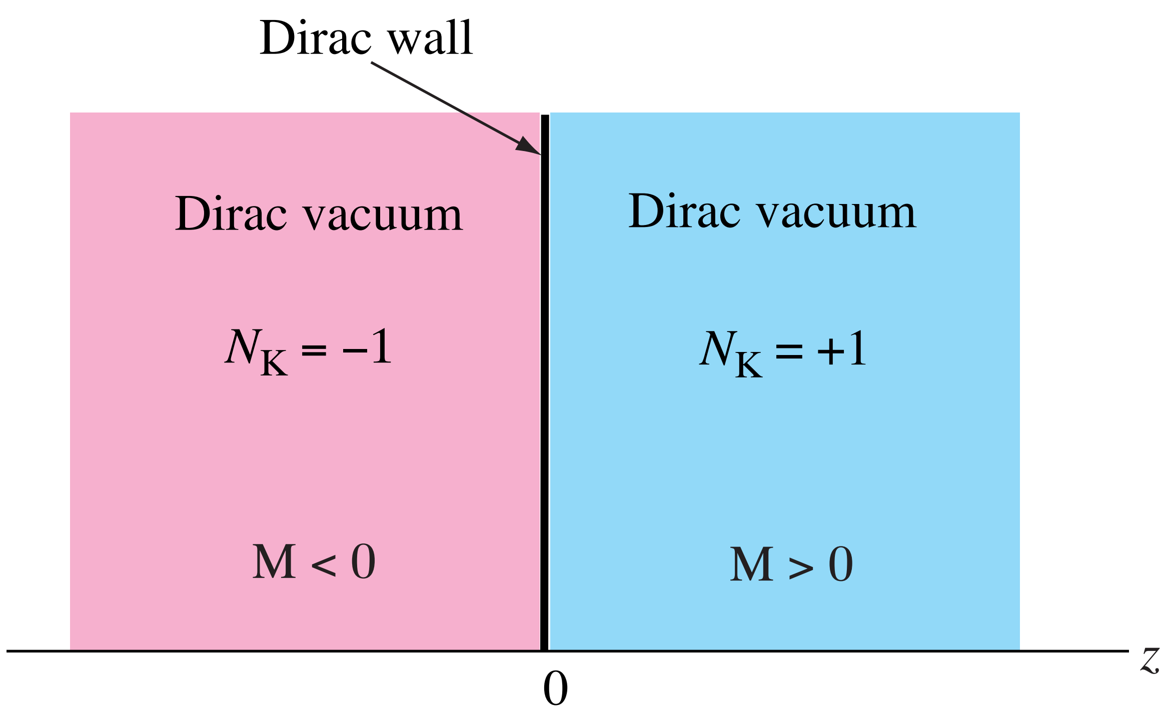
spectrum of Majorana fermion zero modes

$$H_{zm} = c_B \, \hat{\mathbf{z}} \cdot \sigma \times \mathbf{p} = c_B \, (\sigma_x p_y - \sigma_y p_x)$$

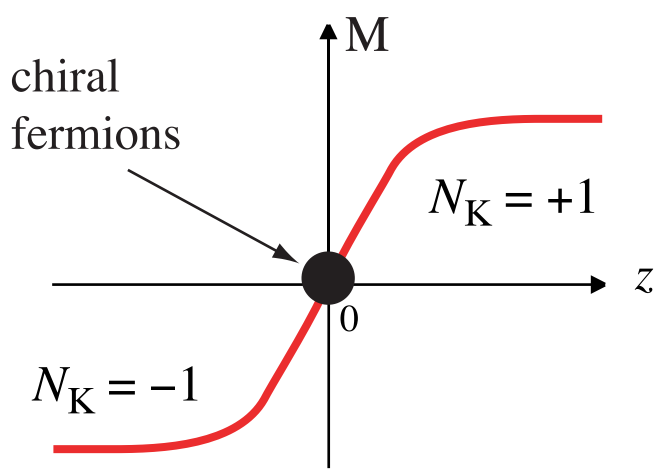
helical fermions

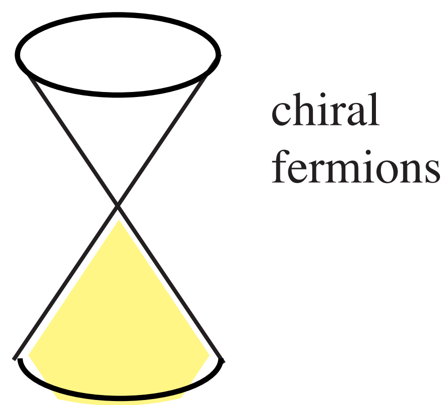


fermion zero modes on Dirac wall



$$H = \begin{pmatrix} -M(z) & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & +M(z) \end{pmatrix}$$

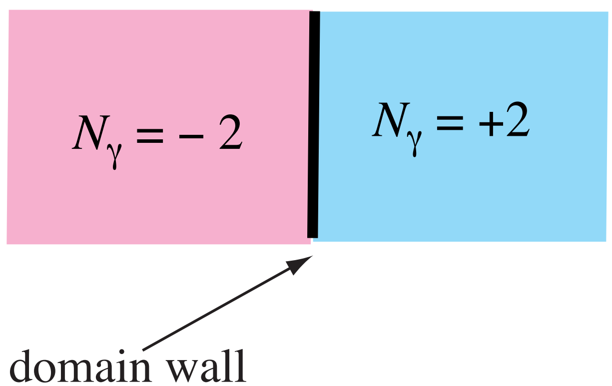


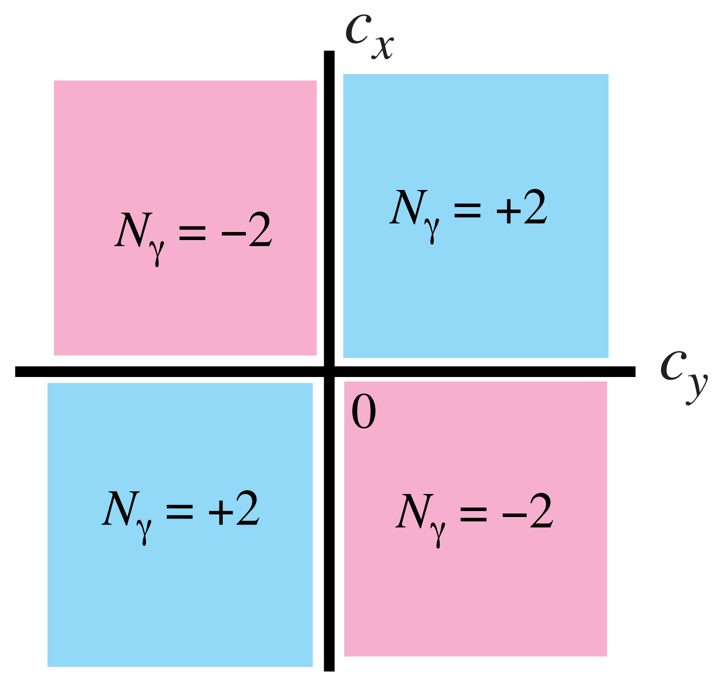


Volkov-Pankratov,
2D massless fermions
in inverted contacts
JETP Lett. **42**, 178 (1985)

Majorana fermions on interface in topological superfluid ^{3}He-B

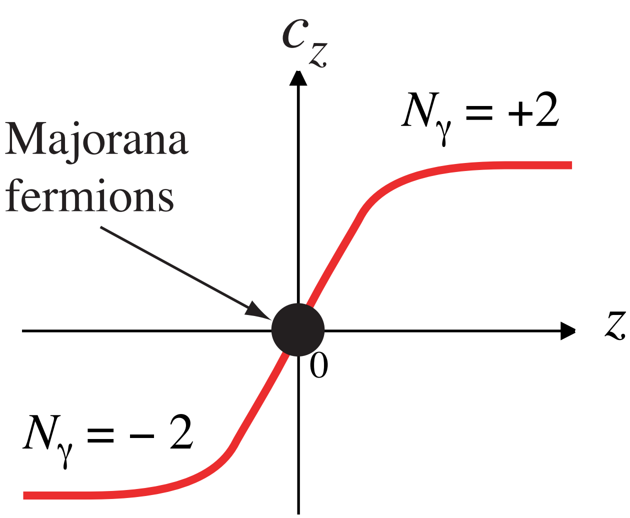
$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z \\ \sigma_x c_x p_x + \sigma_y c_y p_y + \sigma_z c_z p_z & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

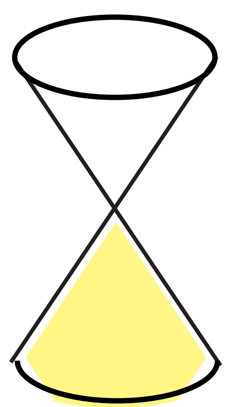




phase diagram

one of 3 "speeds of light" changes sign across wall

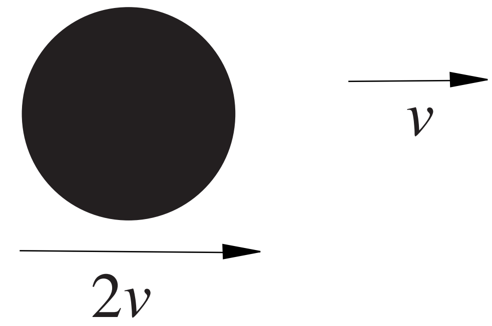




spectrum of fermion zero modes

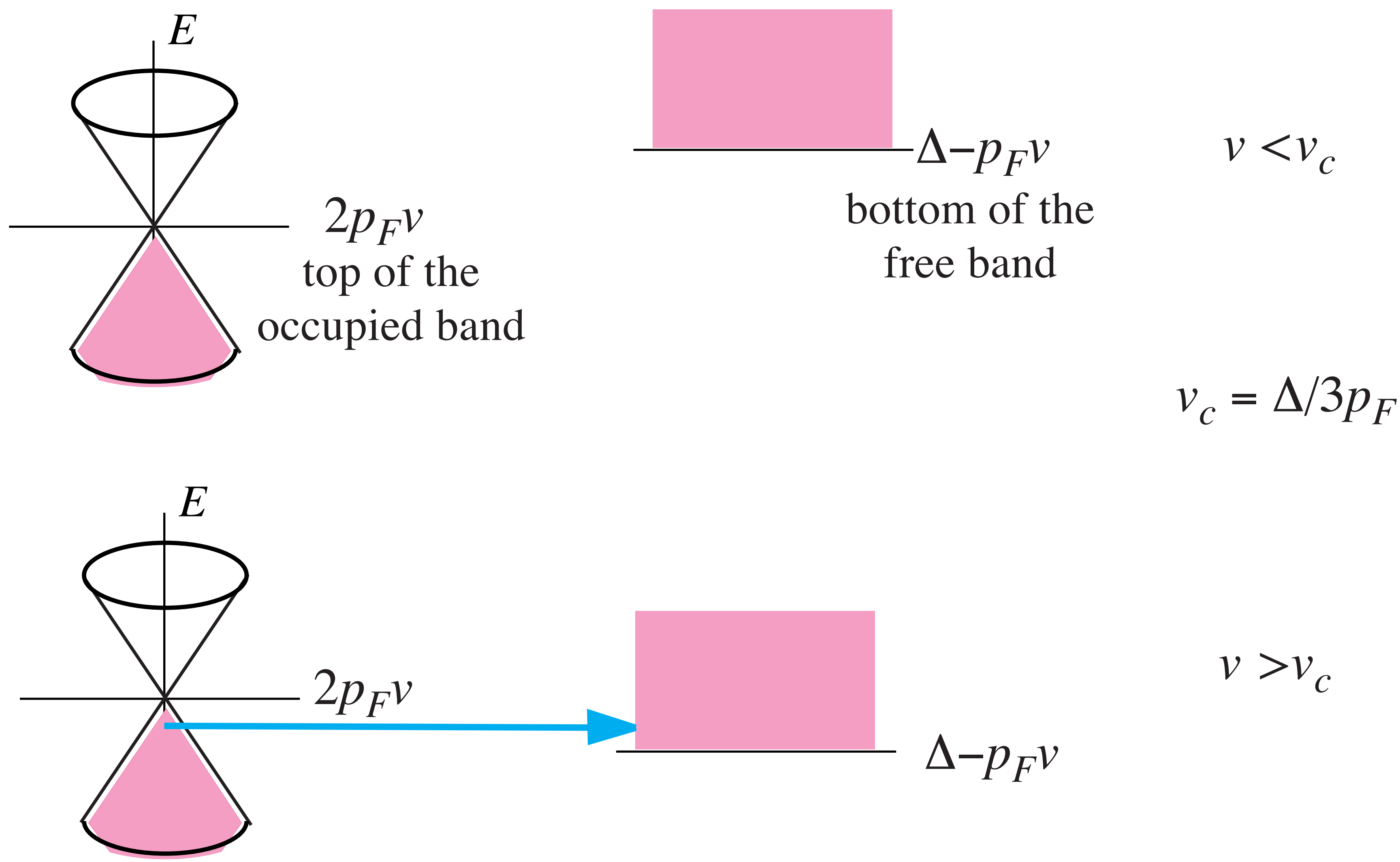
$$H_{zm} = c\,(\sigma_x p_y - \sigma_y p_x)$$

Lancaster experiments:
probing edge states of 3He-B with vibrating wire



spectrum of surface states

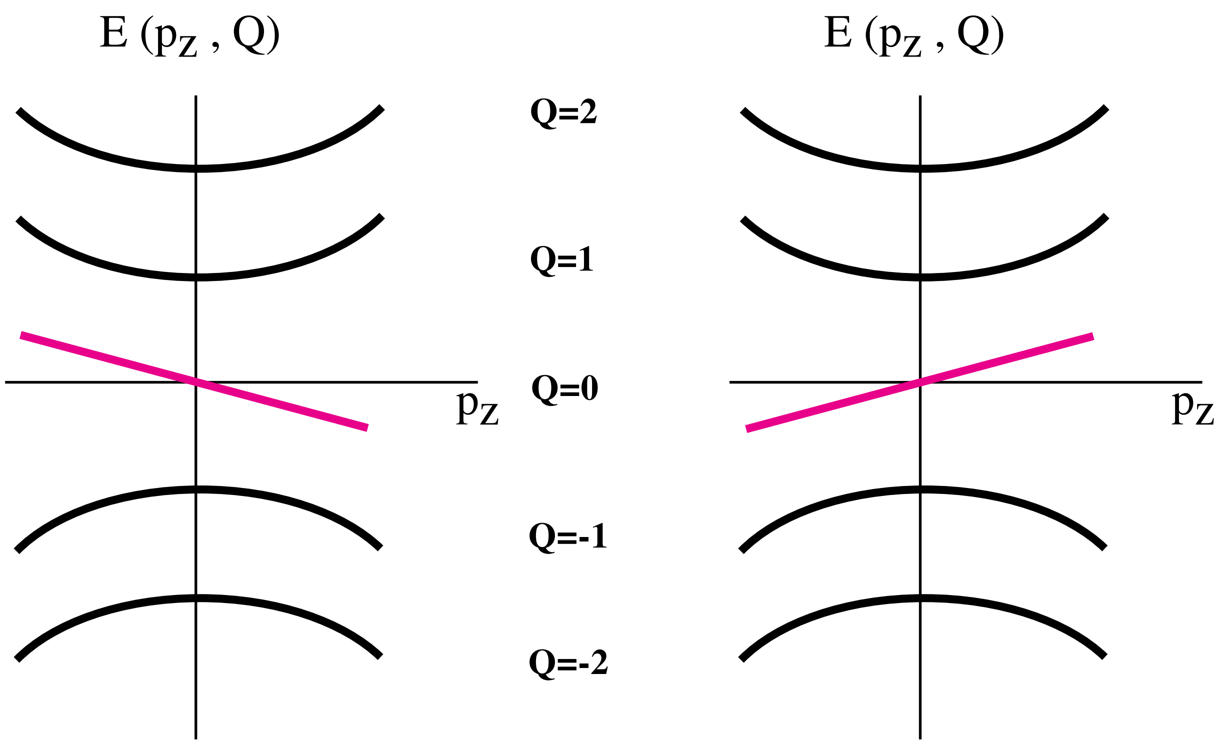
continuous spectrum



Bound states of fermions on cosmic strings and vortices

Spectrum of quarks in core of electroweak cosmic string

quantum numbers: Q - angular momentum & p_z - linear momentum



$E(p_z) = - cp_z$ for d quarks $E(p_z) = cp_z$ for u quark

asymmetric branches cross zero energy

Index theorem:

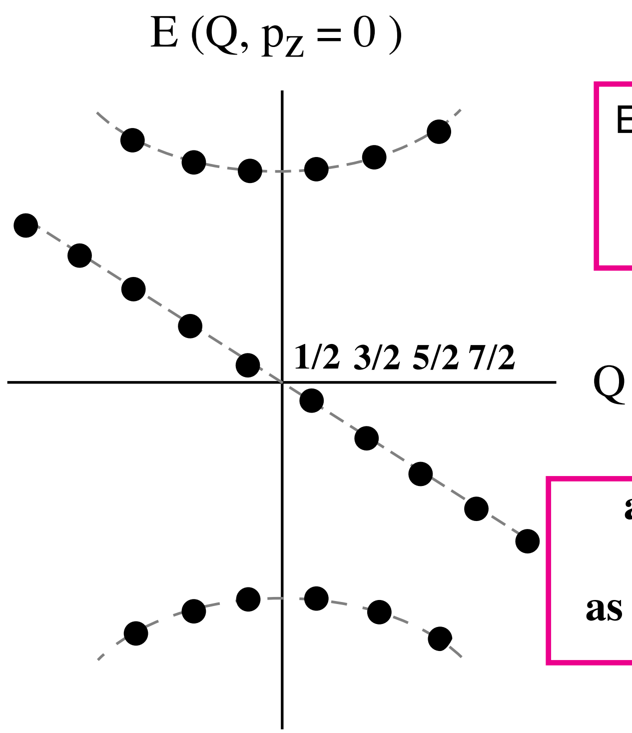
Number of asymmetric branches = N N is vortex winding number

Jackiw & Rossi
Nucl. Phys. B**190**, 681 (1981)

Bound states of fermions on vortex in s-wave superconductor

Caroli, de Gennes & J. Matricon, Phys. Lett. **9** (1964) 307

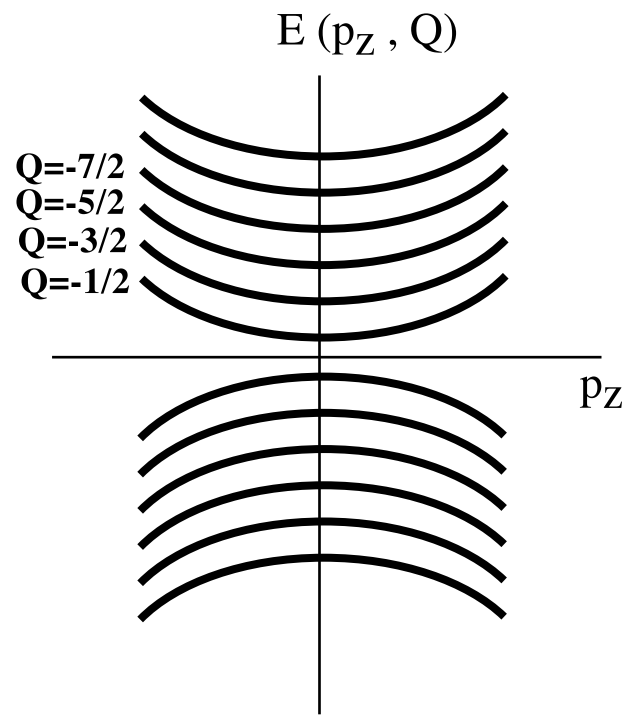
$N_K = 0$



$$E(Q,p_z) = -Q\,\omega_0(p_z)$$

$$\omega_0 = \Delta^2/E_F \ll \Delta$$

asymmetric branch as function of Q



Angular momentum Q is half-odd integer
in s-wave superconductor

no true fermion zero modes:
no asymmetric branch as function of p_z

Index theorem for approximate fermion zero modes:

Number of asymmetric Q-branches = 2N
N is vortex winding number

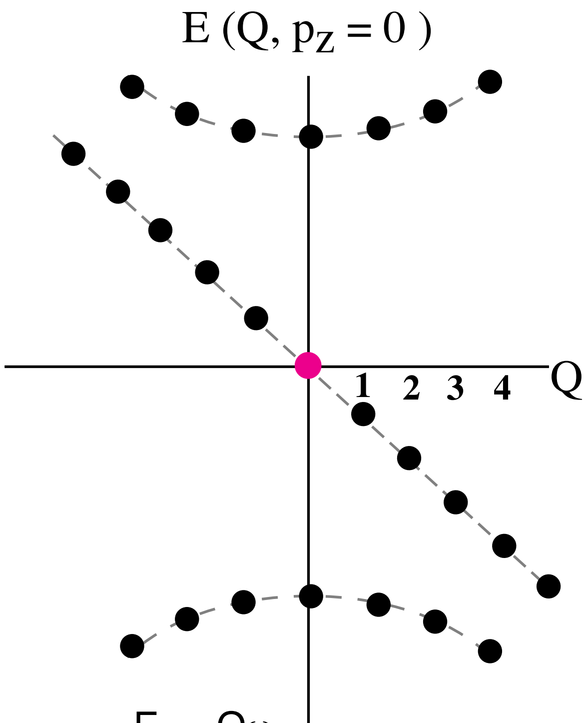
GV JETP Lett. **57**, 244 (1993)

Index theorem for true fermion zero modes?

is the existence of fermion zero modes
related to topology in bulk?

fermions zero modes on symmetric vortex in 3He-B

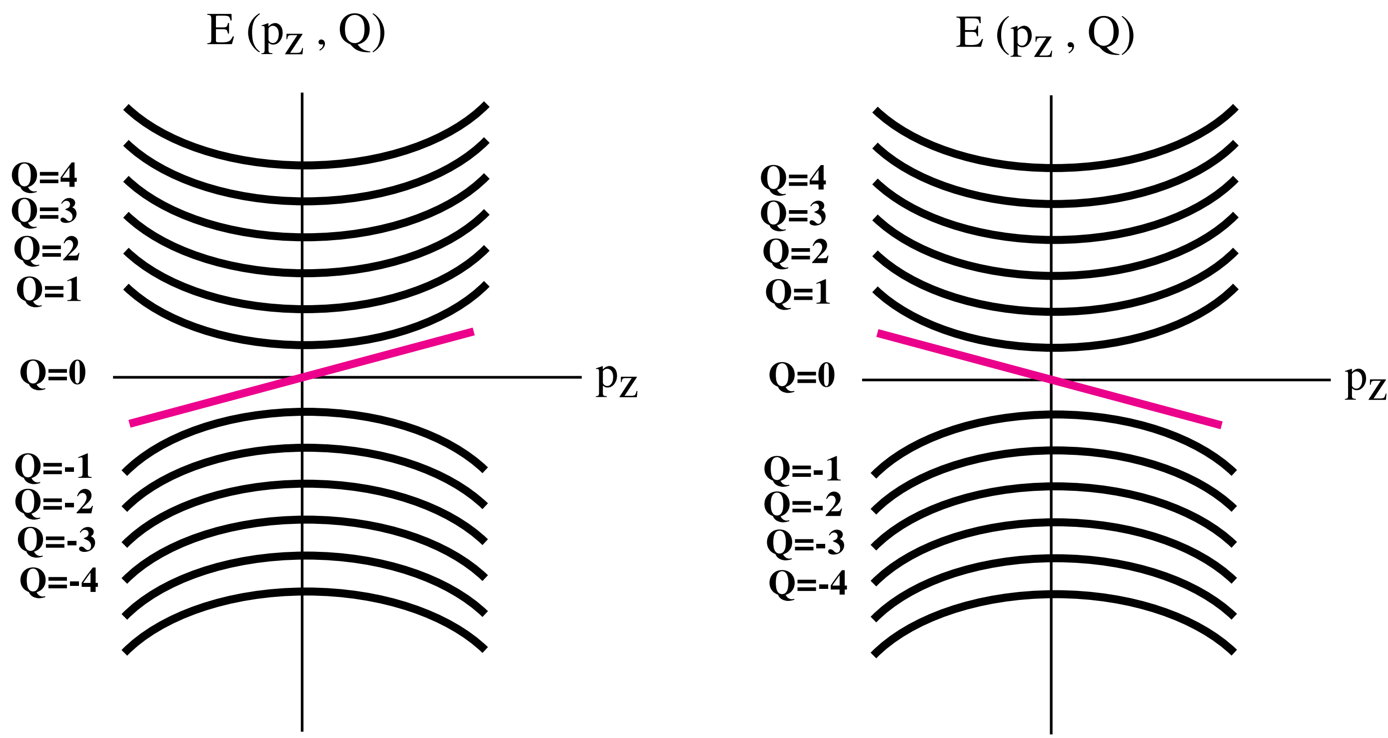
topological ^{3}He-B at $\mu > 0$: $N_\gamma = 2$



$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

Q is integer
for p-wave superfluid ^{3}He-B



helicity + helicity –

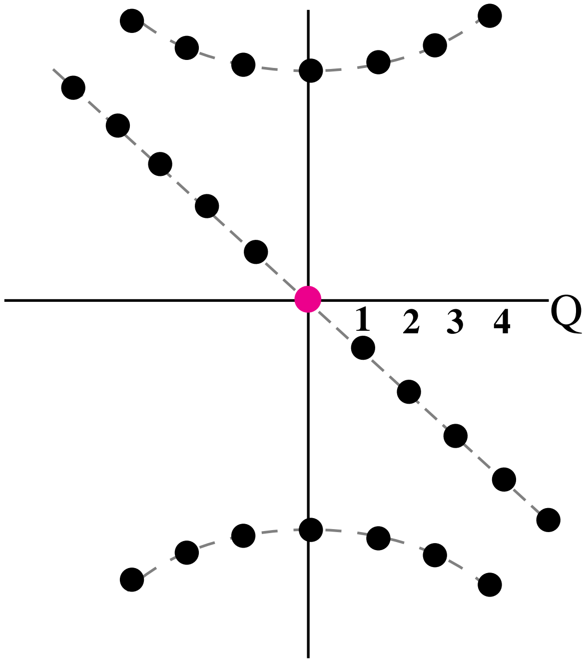
gapless fermions on Q=0 branch form

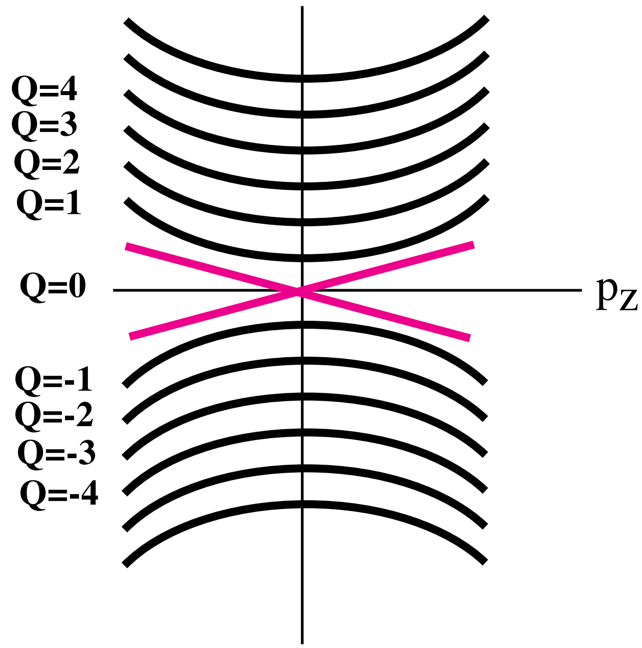
1D Fermi-liquid

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

fermions zero modes on symmetric vortex in 3He-B

topological ^{3}He-B at $\mu > 0$: $N_K = 2$





$E_Q = -Q\omega_0$

$\omega_0 = \Delta^2/E_F << \Delta$

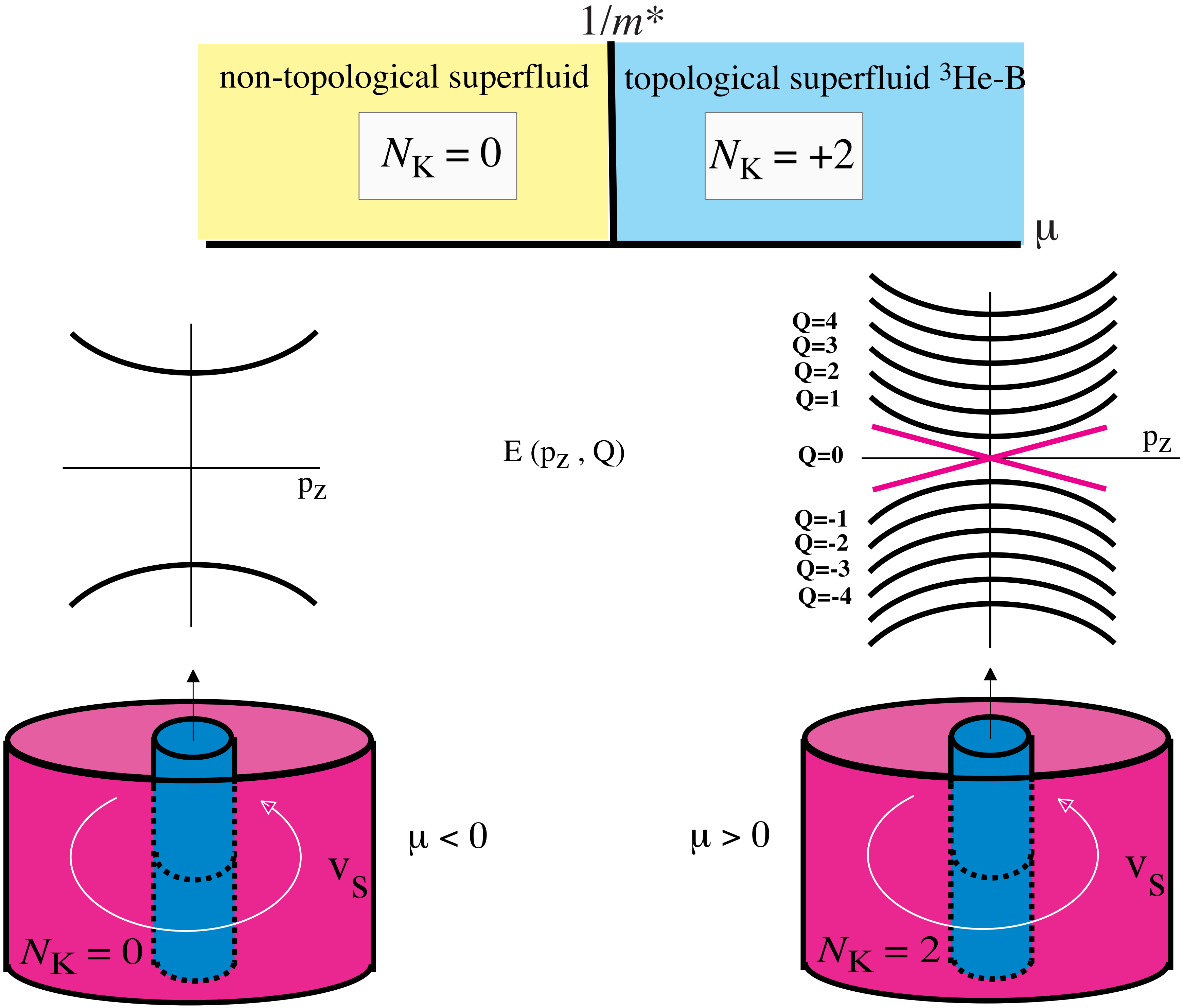
Q is integer
for p-wave superfluid ^{3}He-B

gapless fermions on Q=0 branch form

1D Fermi-liquid

Misirpashaev & GV
Fermion zero modes in symmetric vortices in superfluid 3He,
Physica B **210**, 338 (1995)

topological quantum phase transition in bulk & in vortex core



superfluid ^{3}He-B as non-relativistic limit of relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

relativistic triplet superconductor

$$\downarrow \quad cp << M, \quad \mu << M$$

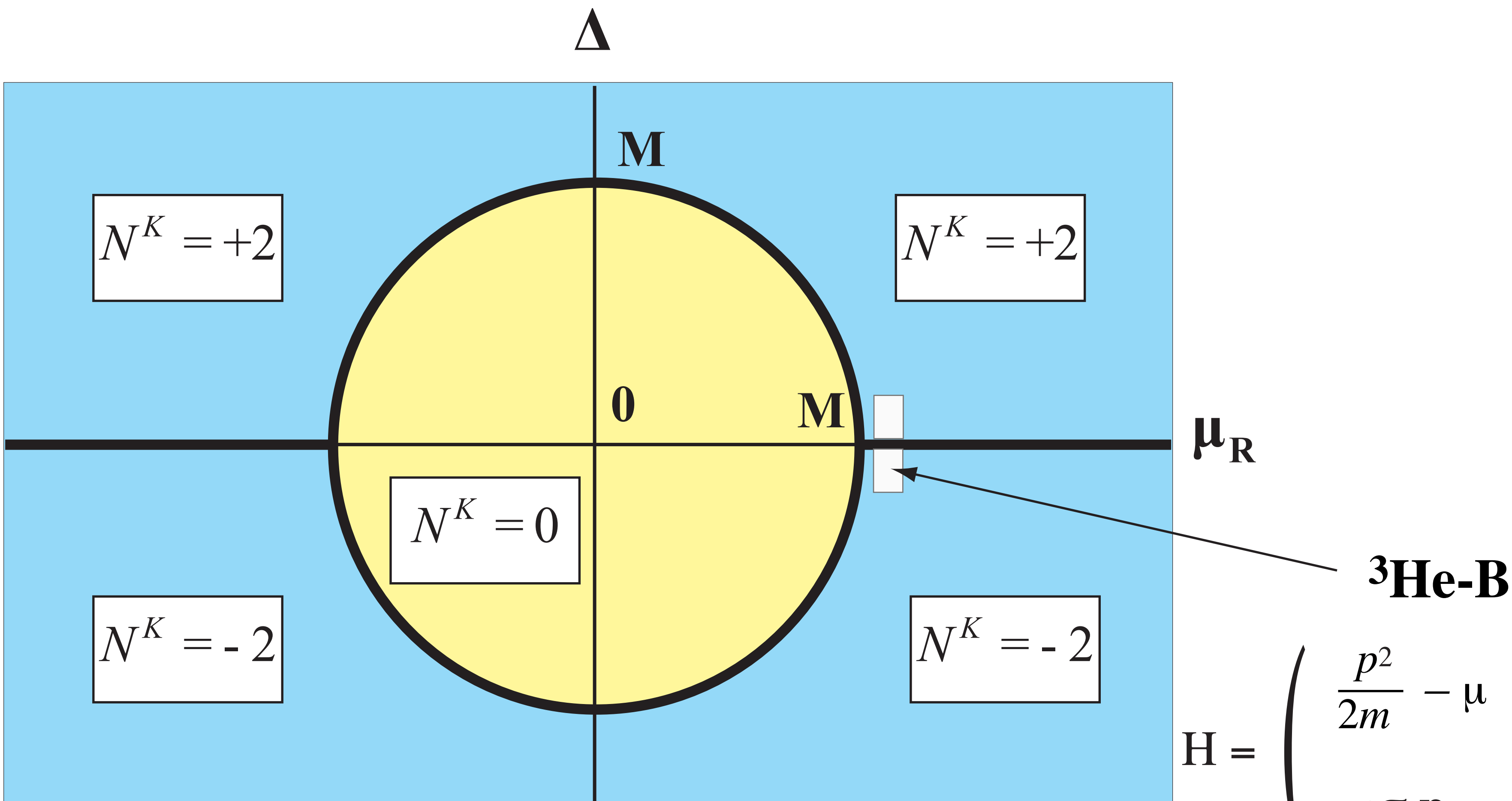
$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c_B\sigma\cdot\mathbf{p} \\ c_B\sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

superfluid ^{3}He-B

$$c_B = c\,\Delta / M \qquad m = M / c^2$$

$$(\mu + M)^2 = \mu_R^2 + \Delta^2$$

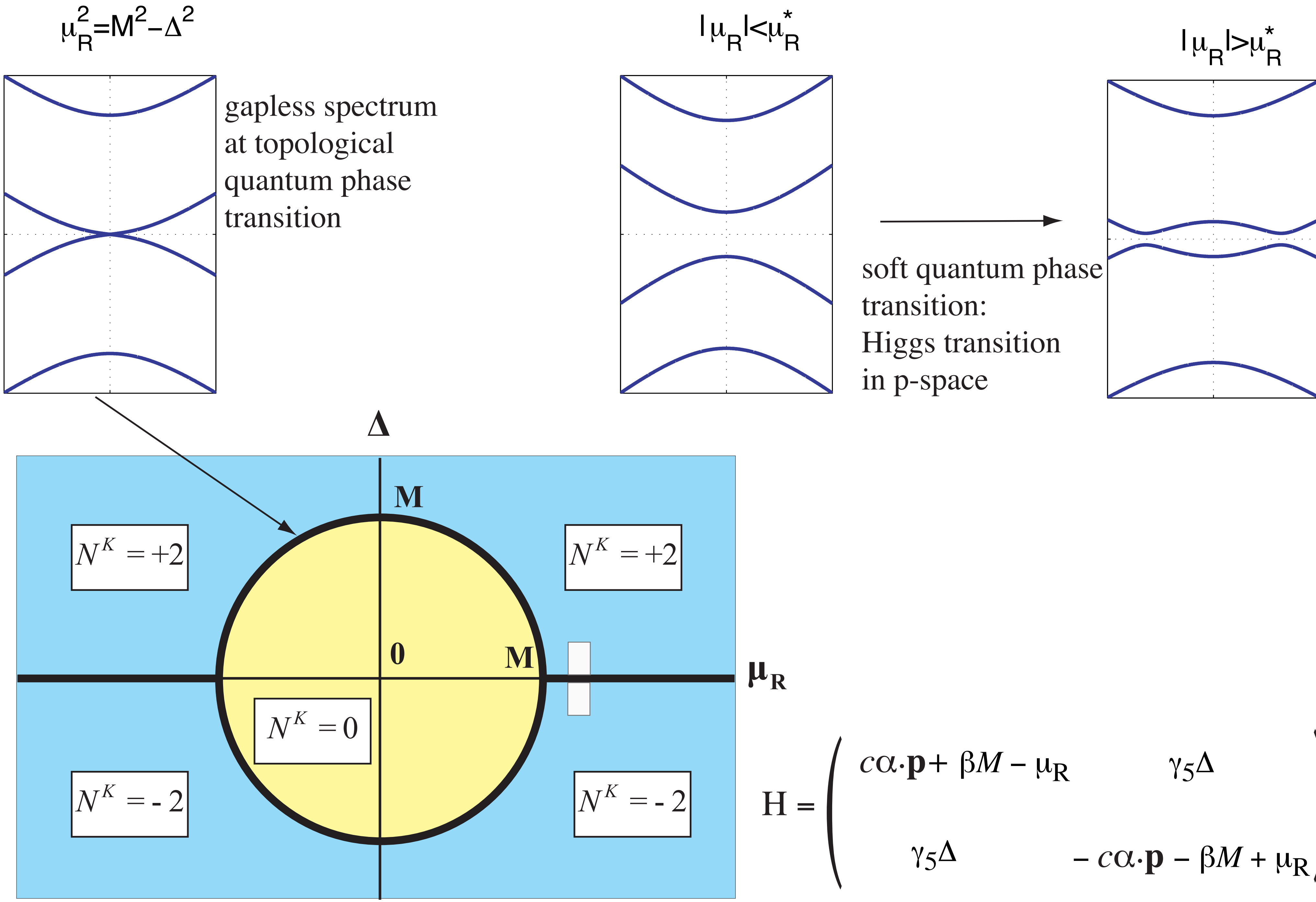
phase diagram of topological states of relativistic triplet superconductor



$$H = \begin{pmatrix} \frac{p^2}{2m} - \mu & c\sigma\cdot\mathbf{p} \\ c\sigma\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu \end{pmatrix}$$

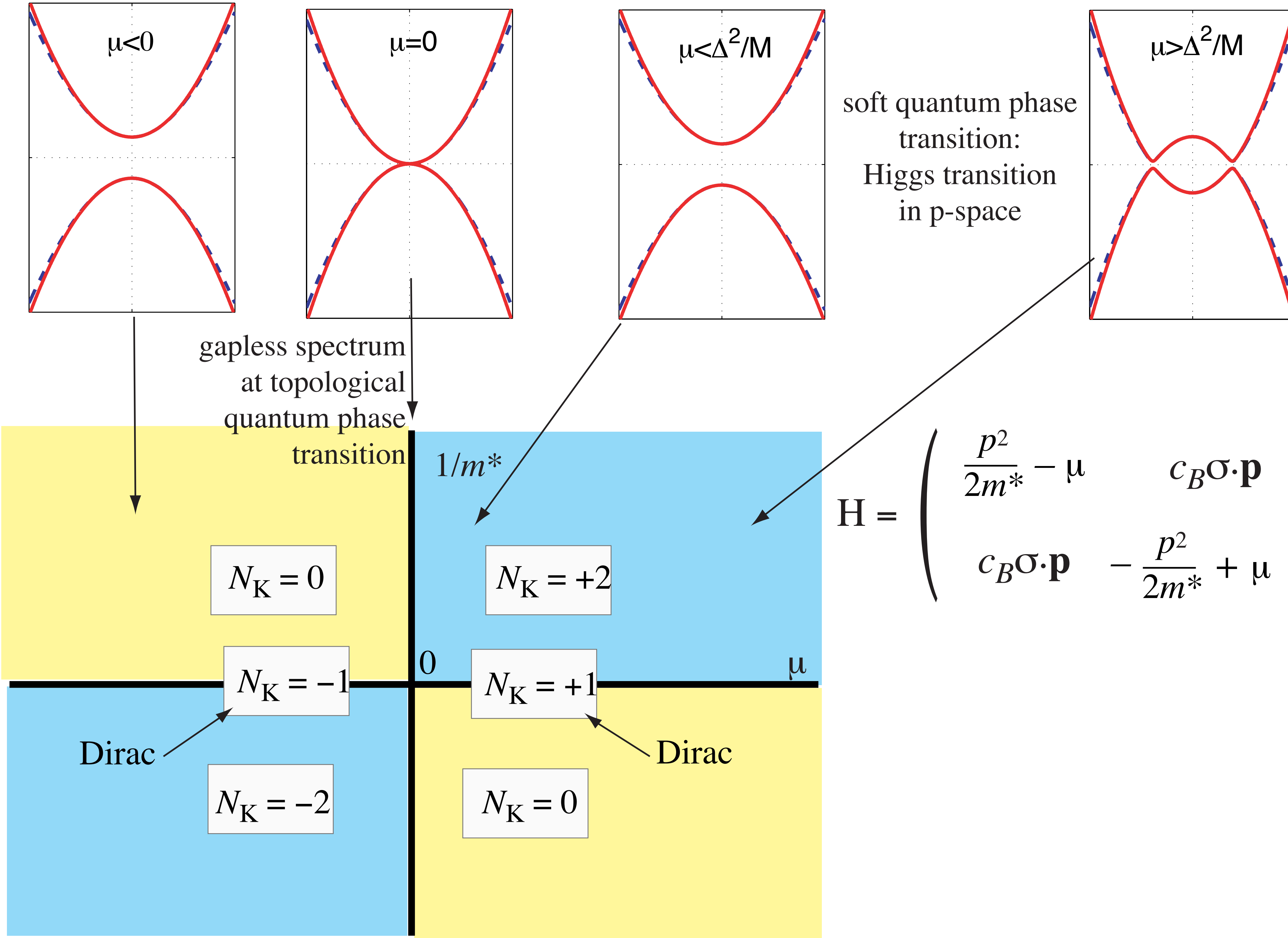
$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

energy spectrum in relativistic triplet superconductor



$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$

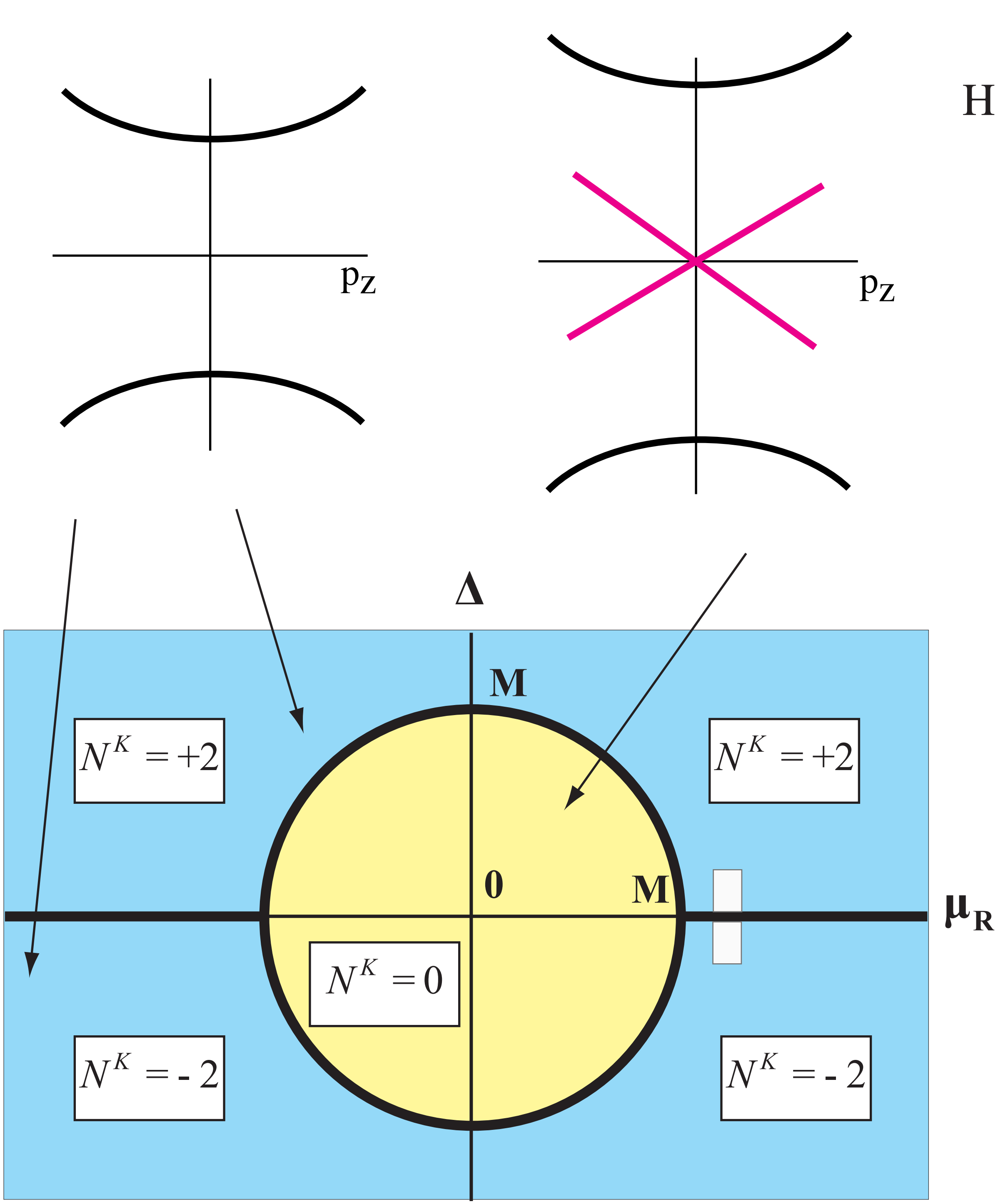
spectrum of non-relativistic ^{3}He-B



$$H = \begin{pmatrix} \frac{p^2}{2m^*} - \mu & c_B \sigma\cdot\mathbf{p} \\ c_B \sigma\cdot\mathbf{p} & -\frac{p^2}{2m^*} + \mu \end{pmatrix}$$

fermion zero modes in relativistic triplet superconductor

$$H = \begin{pmatrix} c\alpha\cdot\mathbf{p} + \beta M - \mu_R & \gamma_5\Delta \\ \gamma_5\Delta & -c\alpha\cdot\mathbf{p} - \beta M + \mu_R \end{pmatrix}$$



vortices in topological superconductors have fermion zero modes

generalized index theorem ?

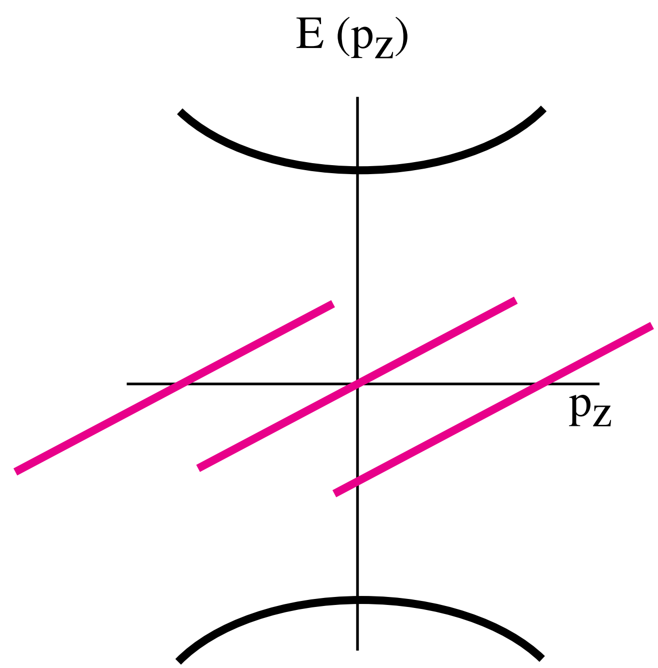
possible index theorem for fermion zero modes on vortices

(interplay of ***r***-space and ***p***-space topologies)

$$N_5 = \frac{1}{4\pi^3 i} \mathbf{tr} \left[\int d^3p \, d\omega \, d\phi \;\; \mathbf{G} \, \boldsymbol{\partial}_\omega \, \mathbf{G}^{-1} \, \mathbf{G} \, \boldsymbol{\partial}_\phi \, \mathbf{G}^{-1} \mathbf{G} \, \boldsymbol{\partial}_{p_x} \, \mathbf{G}^{-1} \mathbf{G} \, \boldsymbol{\partial}_{p_y} \, \mathbf{G}^{-1} \mathbf{G} \, \boldsymbol{\partial}_{p_z} \, \mathbf{G}^{-1} \right]$$

for vortices in Dirac vacuum

$N_5 = N$ **winding number**



Conclusion

Momentum-space topology determines:

universality classes of quantum vacua

effective field theories in these quantum vacua

topological quantum phase transitions (Lifshitz, plateau, etc.)

quantization of Hall and spin-Hall conductivity

topological Chern-Simons & Wess-Zumino terms

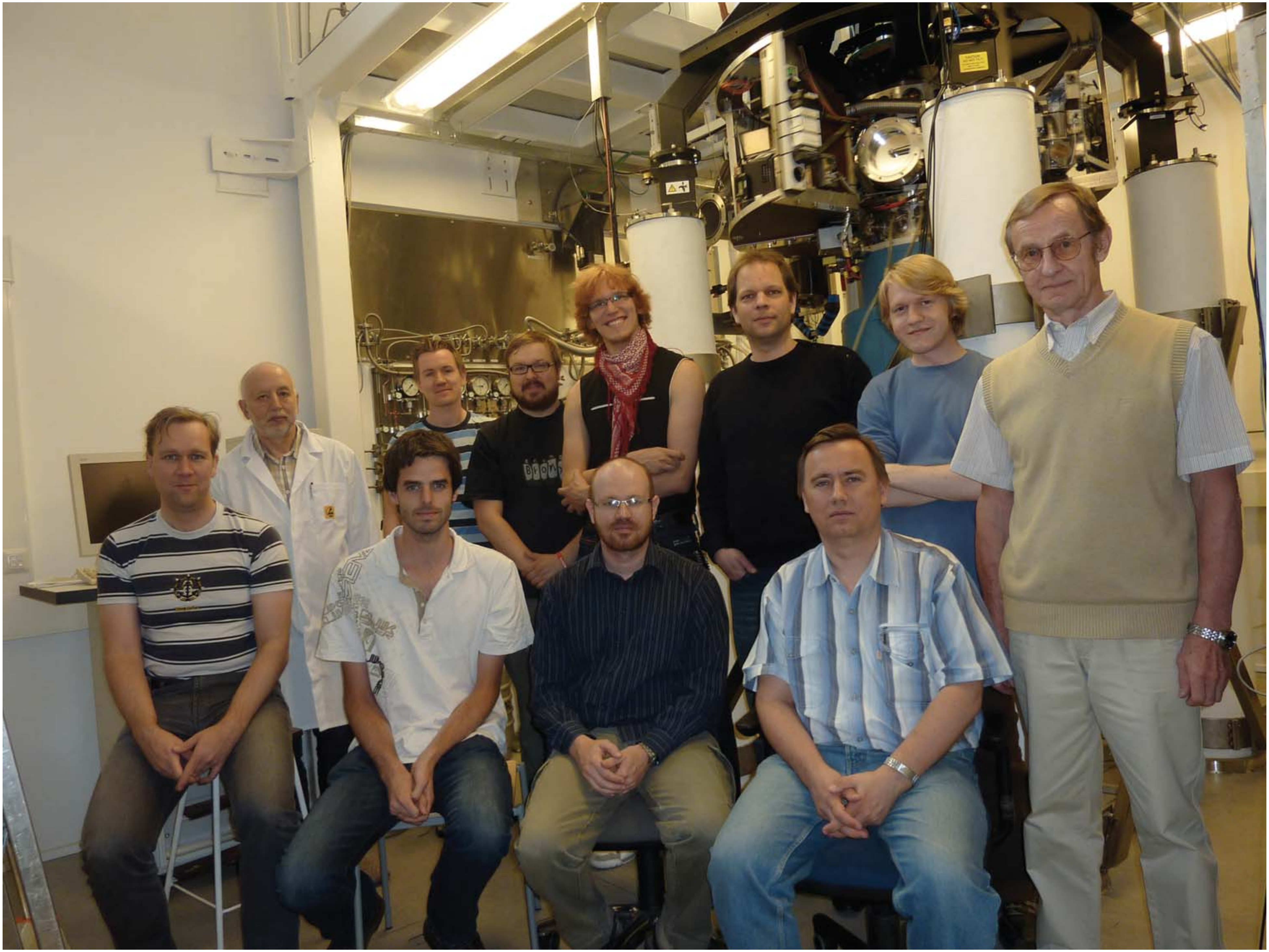
quantum statistics of topological objects

spectrum of edge states & fermion zero modes on walls & quantum vortices

chiral anomaly & vortex dynamics, etc.

problem:
generalized index theorem
(from combined p-r topology ?)

vortices in superfluid ^{3}He-B



vortices in superfluid ^{3}He-B

symmetry breaking phase transitions

$SO(3) \times SO(3) \times U(1) \rightarrow SO(3)$

homotopy group

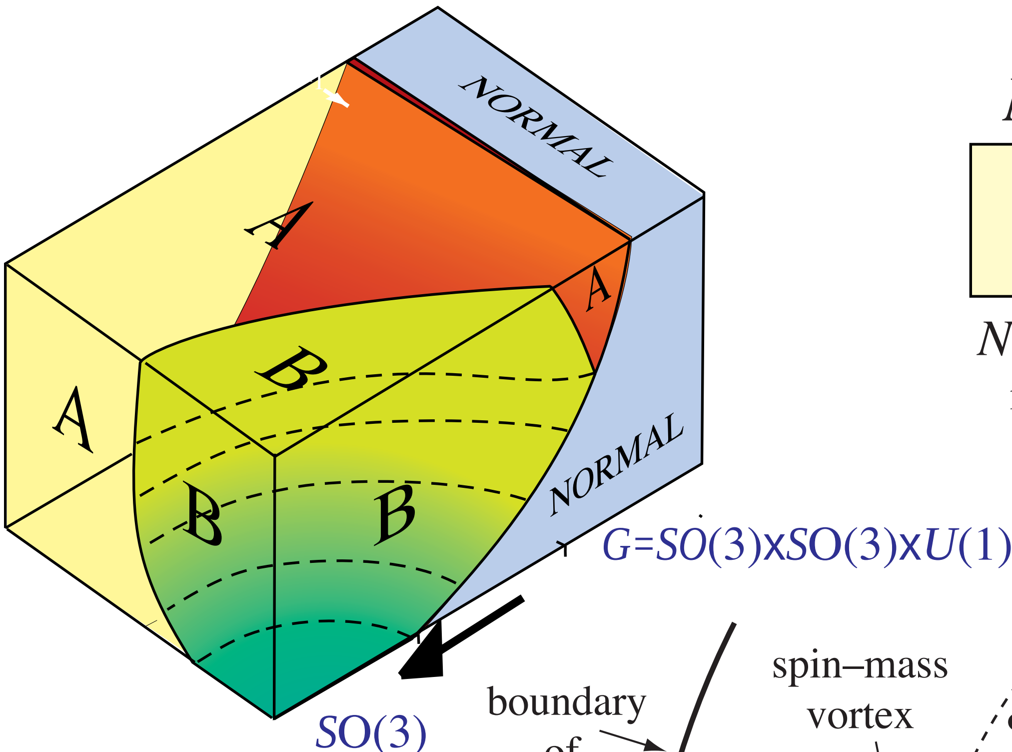
$\pi_1(G/H) = \pi_1(U(1) \times SO(3)) = Z \times Z_2$

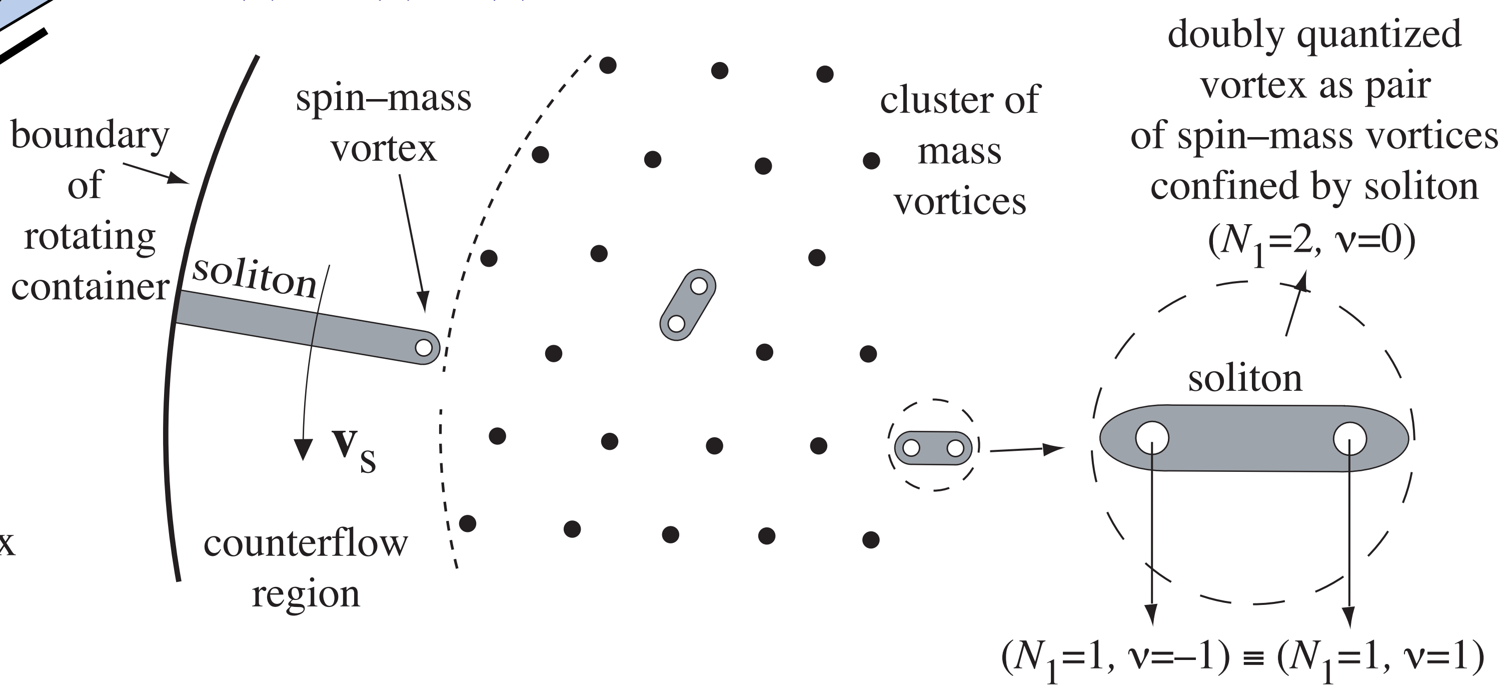
winding numbers

$N_1 = 0, 1, 2, 3 \ldots$ and $\nu = 0, 1$

$1 + 1 = 2$	$1 + 1 = 0$
$N_1 = 1$, $\nu = 0$ mass vortex	$N_1 = 0$, $\nu = 1$ spin vortex

$N_1 = \nu = 1$
spin-mass vortex

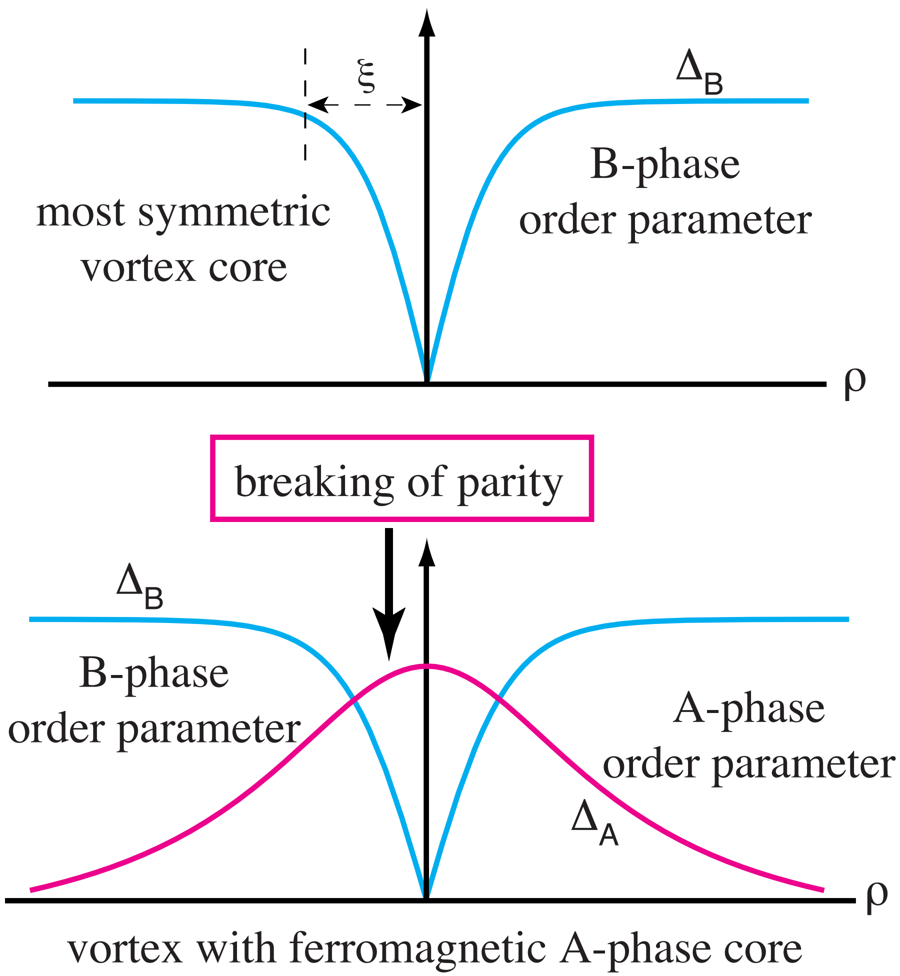




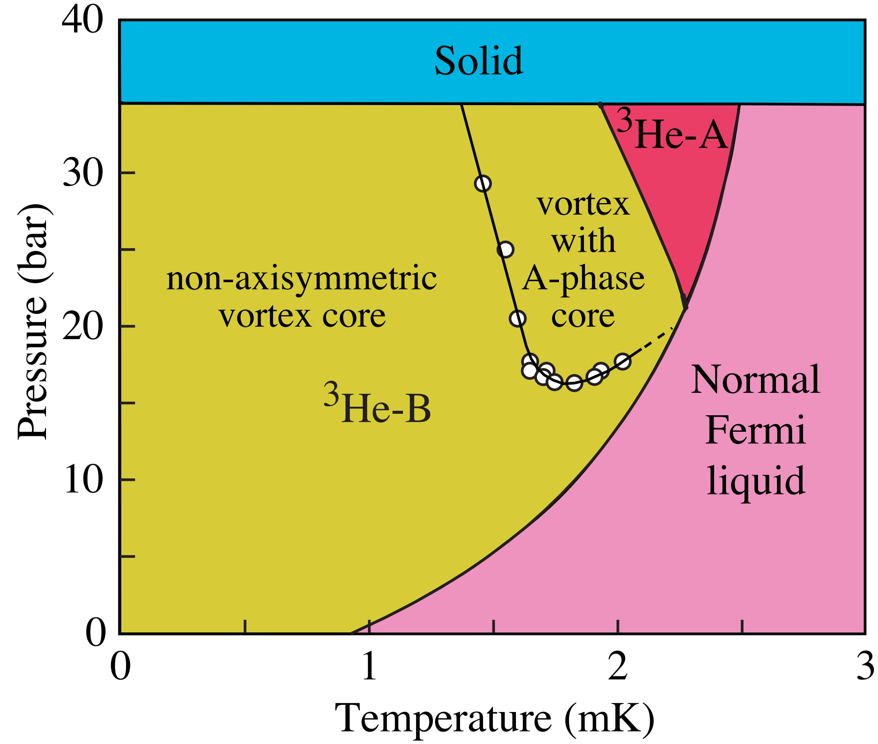
$$\Delta^i_{\alpha\beta} \sim e^{i\Phi} \sigma^k_{\alpha\beta} R^{ki}$$

R^{ki} -- rotation matrix

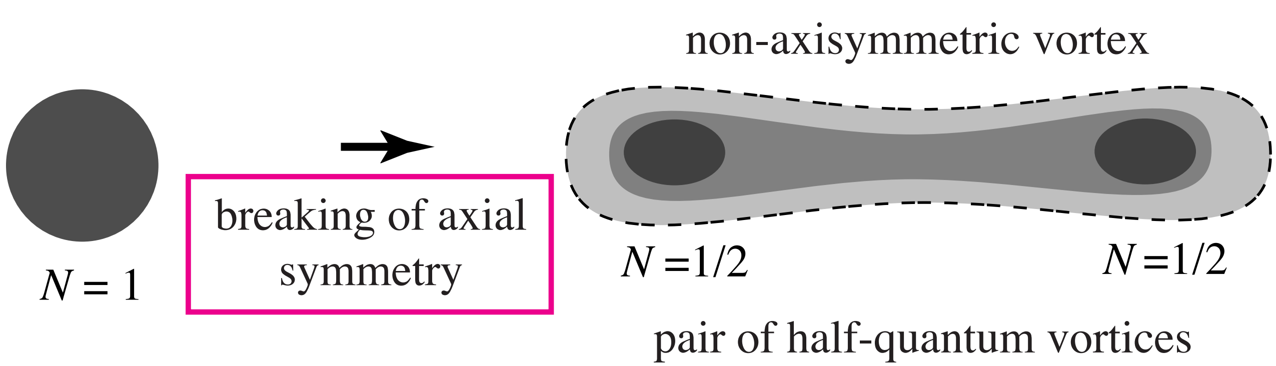
symmetry breaking in the 3He-B vortex core



Phase diagram of first order vortex-core transition in ^{3}He-B

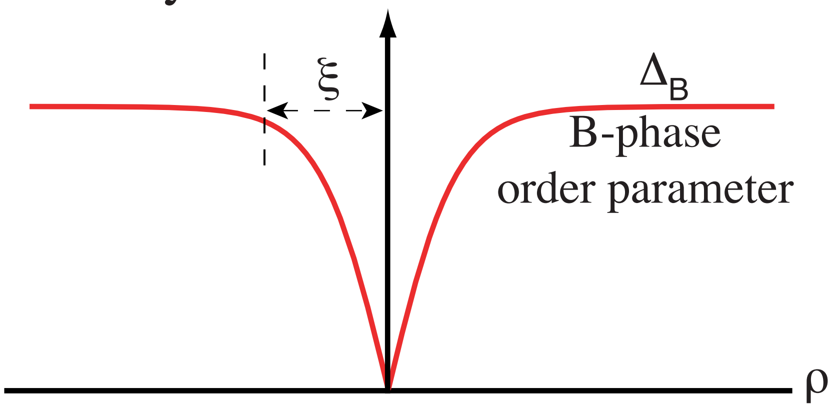


Pekola, Simola, Hakonen, Krusius, et al., PRL **53**, 584 (***1984***)

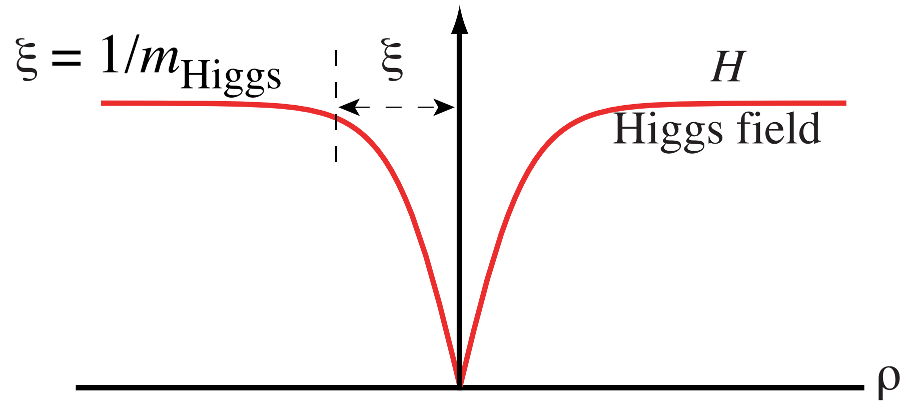


Witten superconducting cosmic string & v-vortex

most symmetric ^{3}He-B vortex

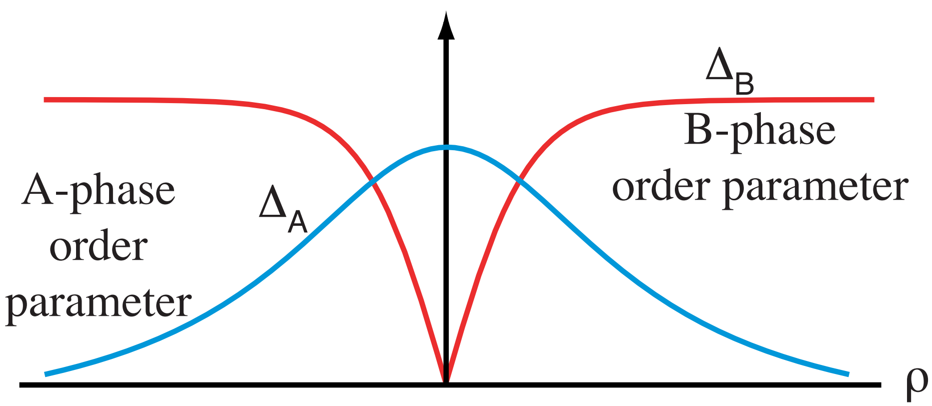


most symmetric cosmic string

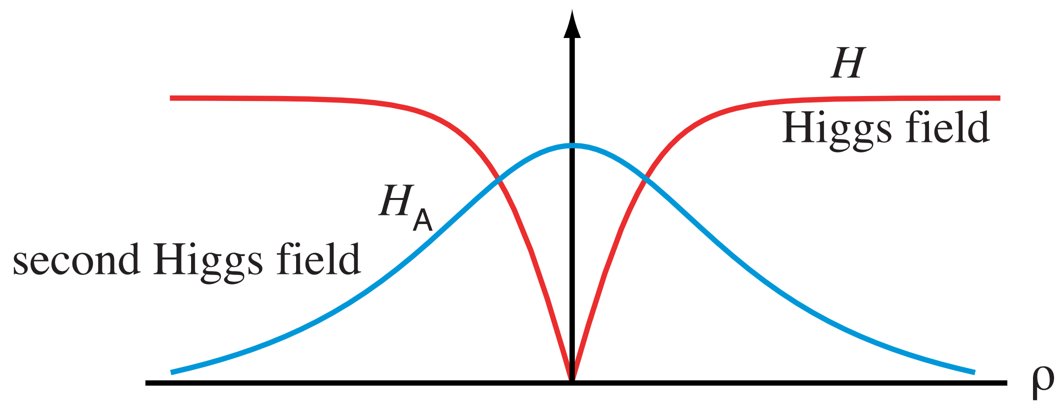


additional breaking of symmetry in the core

v-vortex:
breaking of parity



Witten string:
breaking of electromagnetic symmetry



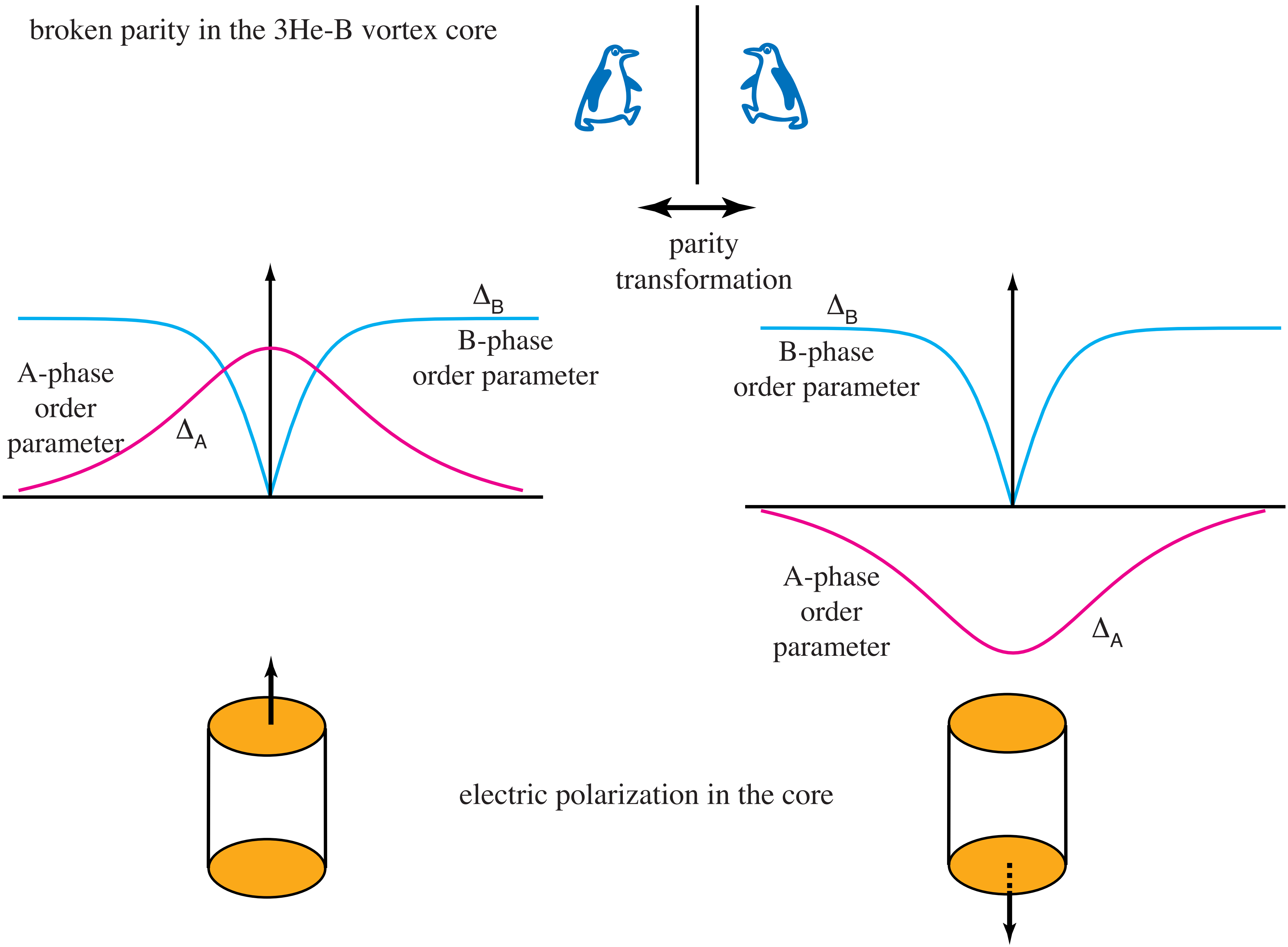
Salomaa & Volovik, PRL **51**, 2040 (1983)

experiment: magnetic core

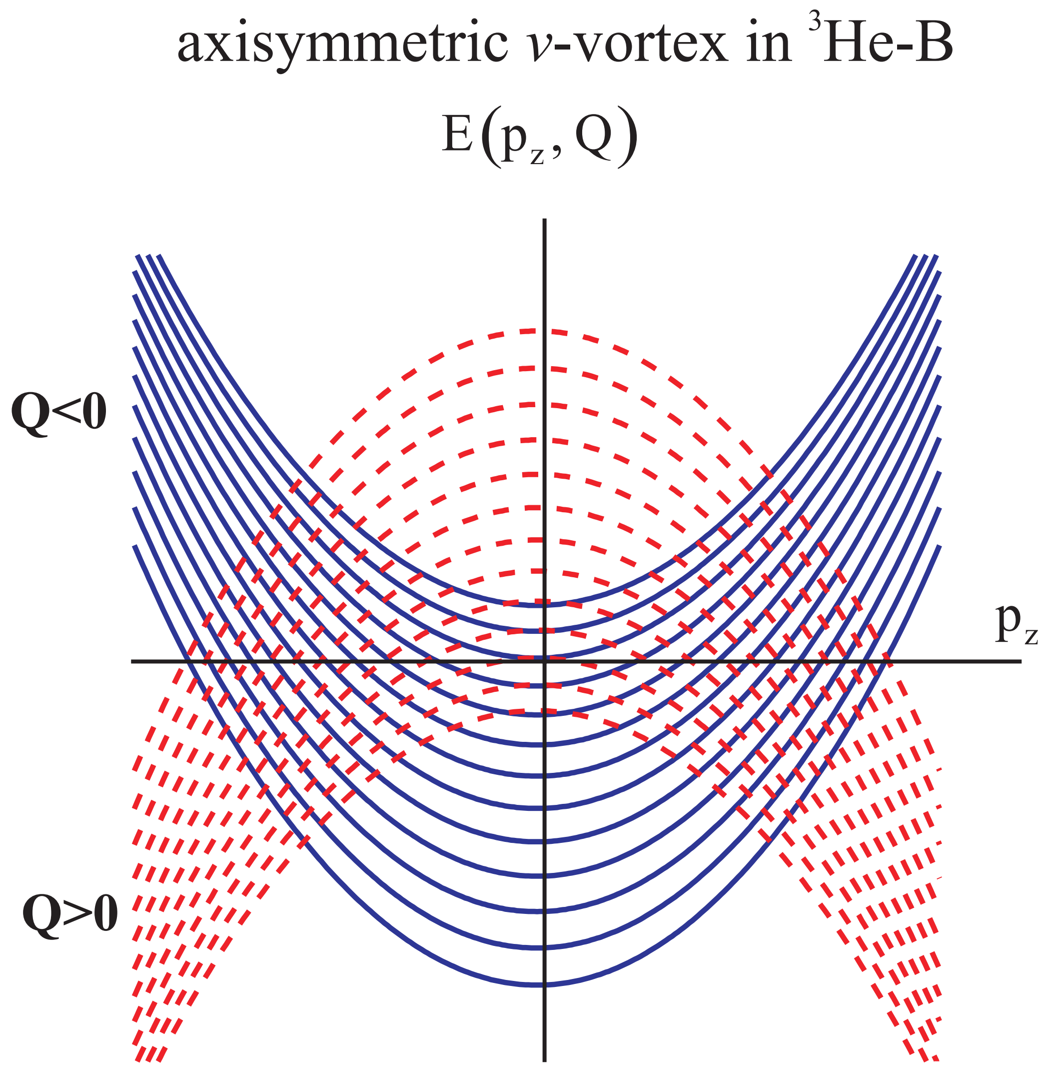
Hakonen, Krusius, Salomaa, et al., PRL **51**, 1362 (1983)

E. Witten, Nucl. Phys. **B 249**, 557 (1985)

superconductivity along the core


broken parity in the 3He-B vortex core
parity
transformation
Δ_B
B-phase
order parameter
A-phase
order
parameter
Δ_A
Δ_B
B-phase
order parameter
A-phase
order
parameter
Δ_A
electric polarization in the core

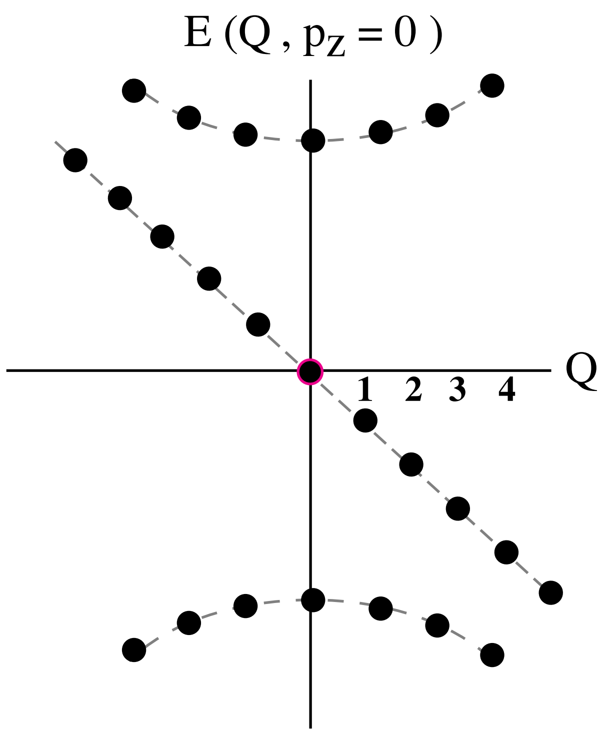
Bound states of fermions on v-vortex in 3He-B

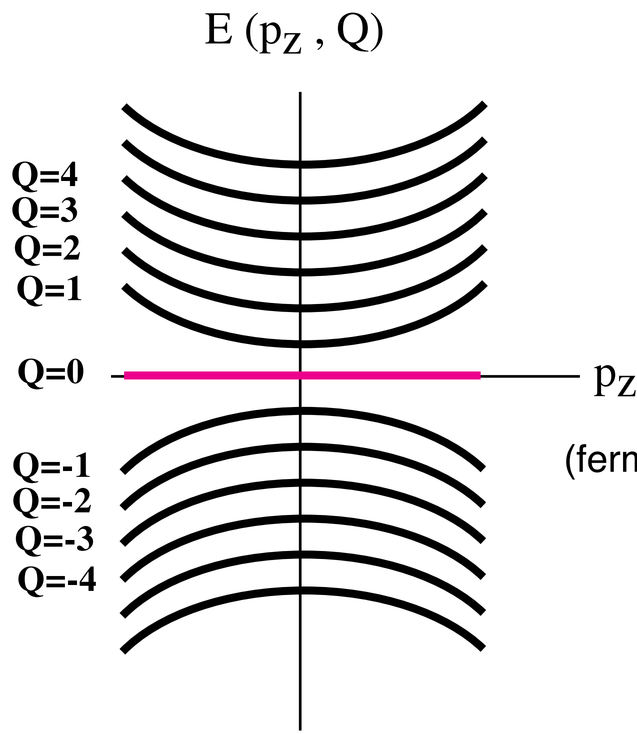


M. A. Silaev, Spectrum of bound fermion states on vortices in 3He-B, JETP Lett. 2009

flat band

zero energy states in symmetric N=1 ^{3}He-A vortex





Fermions on Q=0 branch form

flat Fermi-band

(fermionic condensate or Khodel state)

Kopnin & Salomaa, PRB **44**, 9667 (1991)

forces on vortices:

three nondissipative forces + friction force acting on a vortex line

Iordanskii force

Gravitational Aharonov-Bohm effect

heat bath velocity $\mathbf{v}_n$

Kopnin force

Axial anomaly

$$\mathbf{F}_{\text{Iordanskii}} = \kappa \times \rho_n(\mathbf{v}_s - \mathbf{v}_n)$$

Aharonov-Bohm scattering of quasiparticles on a vortex

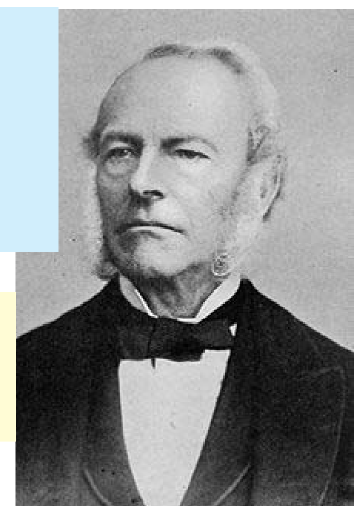
$$\mathbf{F}_{\text{Kopnin}} = \kappa \times \mathbf{C}(T)(\mathbf{v}_n - \mathbf{v}_L)$$

momentum transfer from negative energy states in the core to heat bath analog of baryogenesis

vacuum velocity $\mathbf{v}_s$

$$\mathbf{F}_{\text{Magnus}} = \kappa \times \rho(\mathbf{v}_L - \mathbf{v}_s)$$

vortex velocity $\mathbf{v}_L$



momentum transfer between vortex and superfluid vacuum

Magnus–Joukowski lifting force in classical hydrodynamics

Stokes friction force

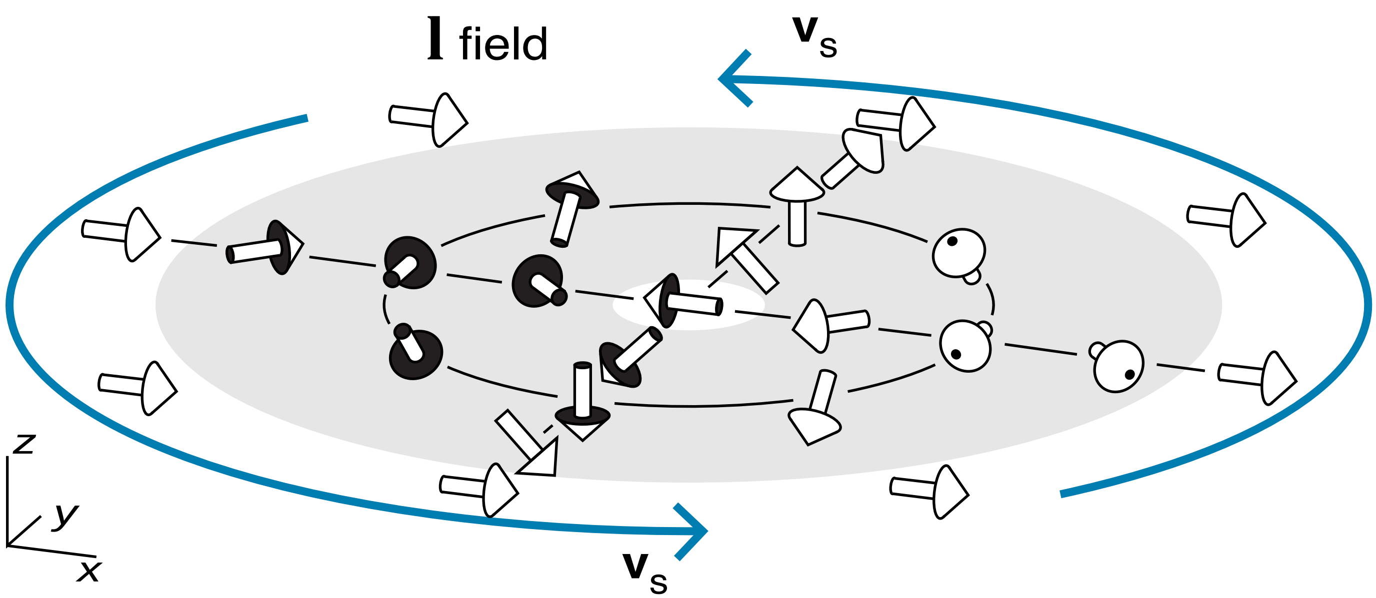
$$\mathbf{F}_{\text{Magnus}} + \mathbf{F}_{\text{Iordanskii}} + \mathbf{F}_{\text{Kopnin}} + \mathbf{F}_{\text{Stokes}} = 0$$

$$\mathbf{F}_{\text{Stokes}} = -\gamma\,(\mathbf{v}_L - \mathbf{v}_n)$$

Momentogenesis by N=2 vortex-skyrmion

$$m = (1/4\pi)\iint dx\,dy\;\left(\mathbf{l}\cdot\left(\partial\mathbf{l}/\partial x \times \partial\mathbf{l}/\partial y\right)\right) = 1$$

vortex-skyrmion with N=2m=2 circulation quanta



l=l(r-vt)

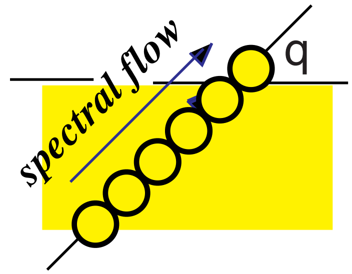
Momentum transfer from vacuum to the heat bath (matter) gives extra topological force on skyrmion (spectral-flow force)

$$\mathbf{F} = \int d^3r\;\dot{\mathbf{P}} = (1/2\pi^2)\int d^3r\;\left(\mathbf{B}\cdot\mathbf{E}\right) p_F\,\mathbf{l} = (1/2\pi^2)\,\hbar\, p_F^3 \int d^3r\;\left(\nabla\times\mathbf{l}\cdot\; d\mathbf{l}/dt\right)\mathbf{l}$$

$$= 2\pi\hbar\,(1/3\pi^2)\,p_F^3\;\hat{\mathbf{z}}\times(\mathbf{v}_n - \mathbf{v}_L)$$

Chiral anomaly:

matter-antimatter asymmetry of Universe *(baryon asymmetry) and Kopnin force*



momentum from vacuum of fermion zero modes

spectral flow produces

baryons from vacuum

$$\dot{\mathbf{P}} = \sum \mathbf{P}_a \dot{n}_a$$

$\mathbf{P}_a$ -- **momentum (fermionic charge)**

e_a -- **effective electric charge**

$$\dot{\mathbf{P}} = (1/4\pi^2)\, \mathbf{B} \cdot \mathbf{E} \sum_a \mathbf{P}_a C_a e_a^2$$

applied to ^{3}He-A

C_a = +1 **for right**

-1 **for left**

$$\dot{B} = \sum B_a \dot{n}_a$$

B_a -- **baryonic charge**

Y_a -- **hypercharge**

$$\dot{B} = (1/4\pi^2)\, \mathbf{B}_Y \cdot \mathbf{E}_Y \sum_a B_a C_a Y_a^2$$

applied to Standard Model

C_a = +1 **for right**

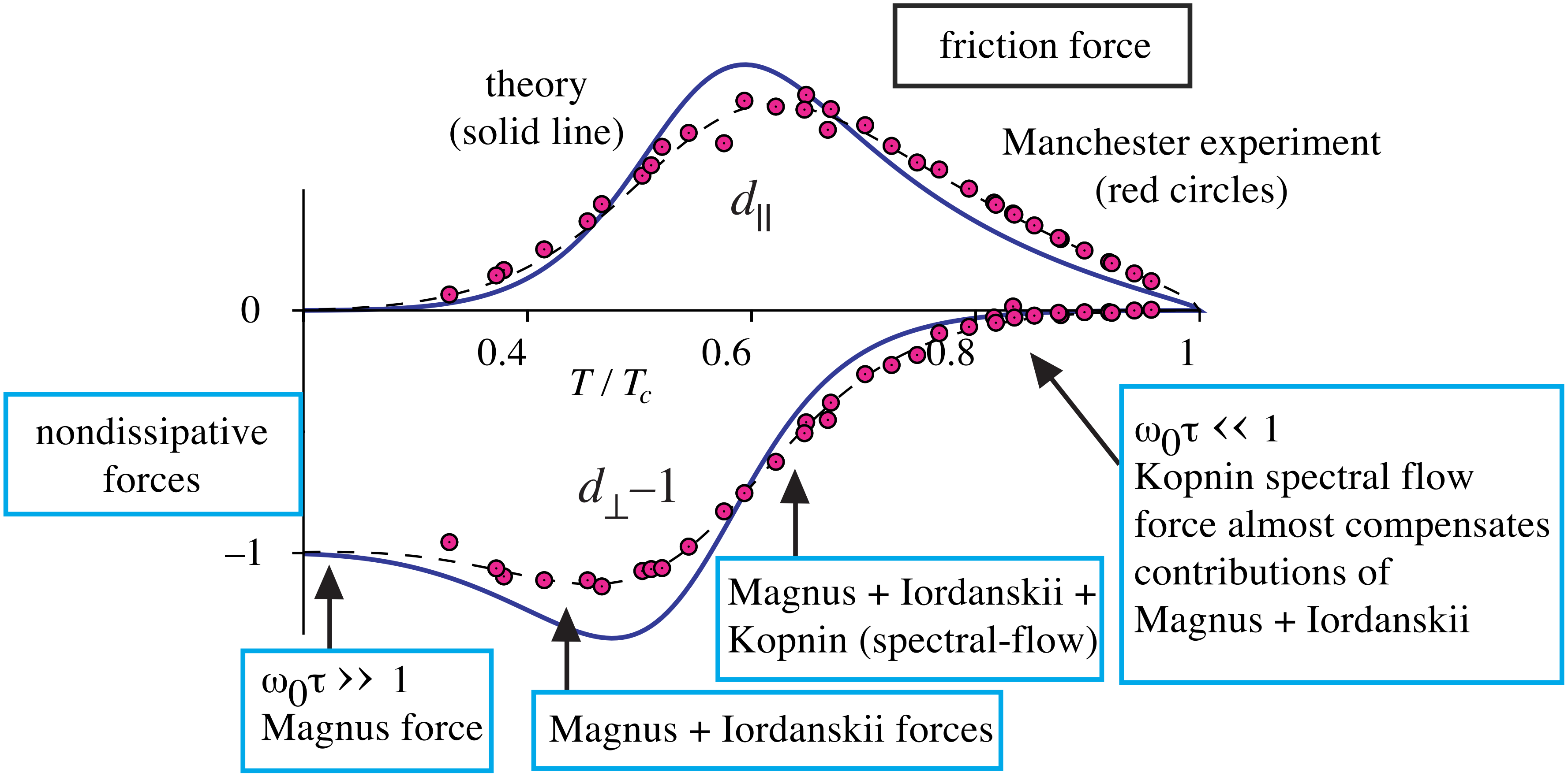
-1 **for left**

chiral anomaly equation (Adler, Bell, Jackiw)

quasiparticles move from vacuum to the positive energy world, where they are scattered by quasiparticles in bulk and transfer momentum from vortex to normal component

this is the source of Kopnin spectral flow force

Experimental and theoretical forces on a vortex



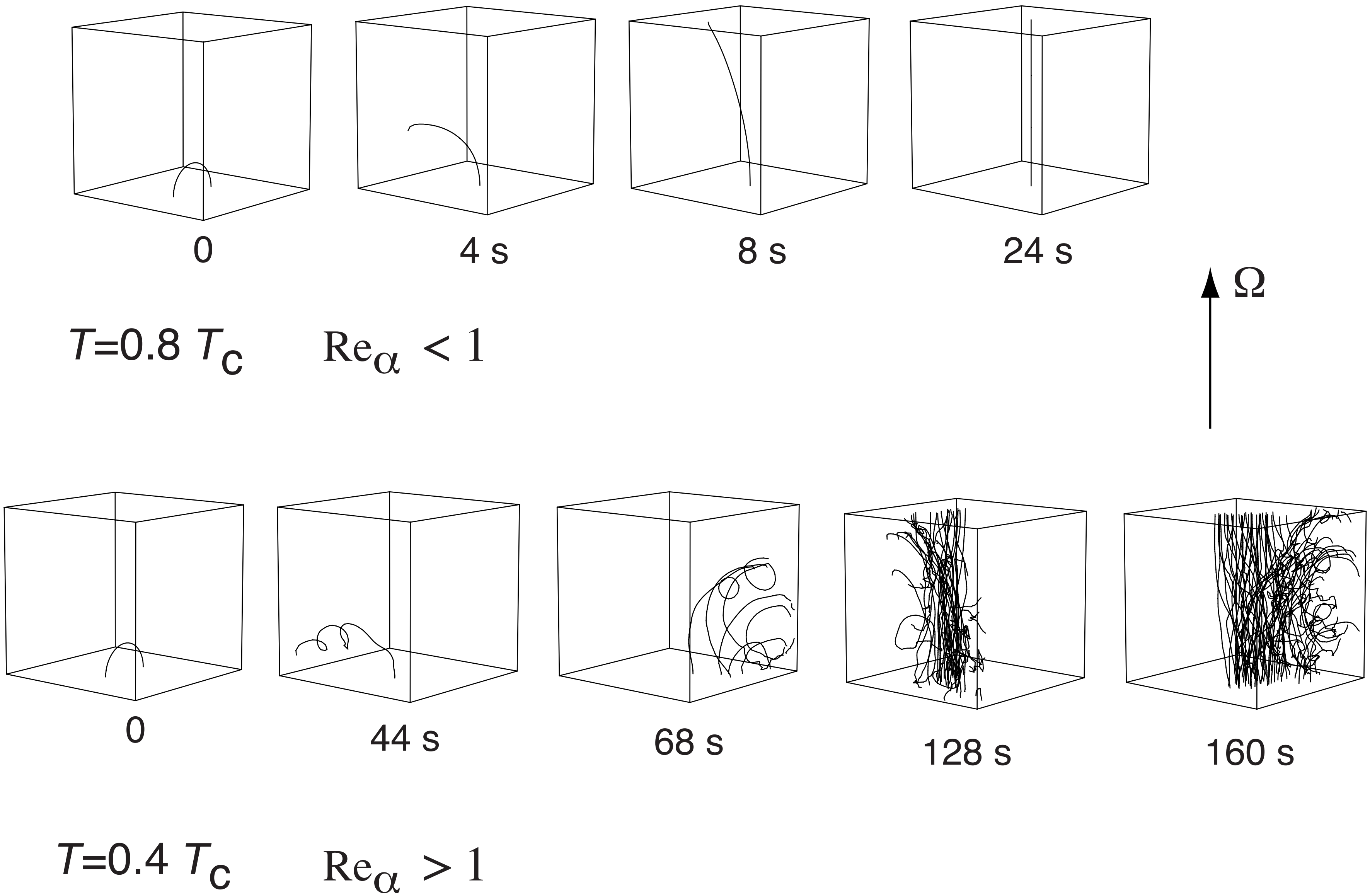
superfluid Reynolds number

$$\mathrm{Re}_{\alpha}(T) = \frac{1 - d_{\perp}}{d_{\|}} \approx \omega_0\tau$$

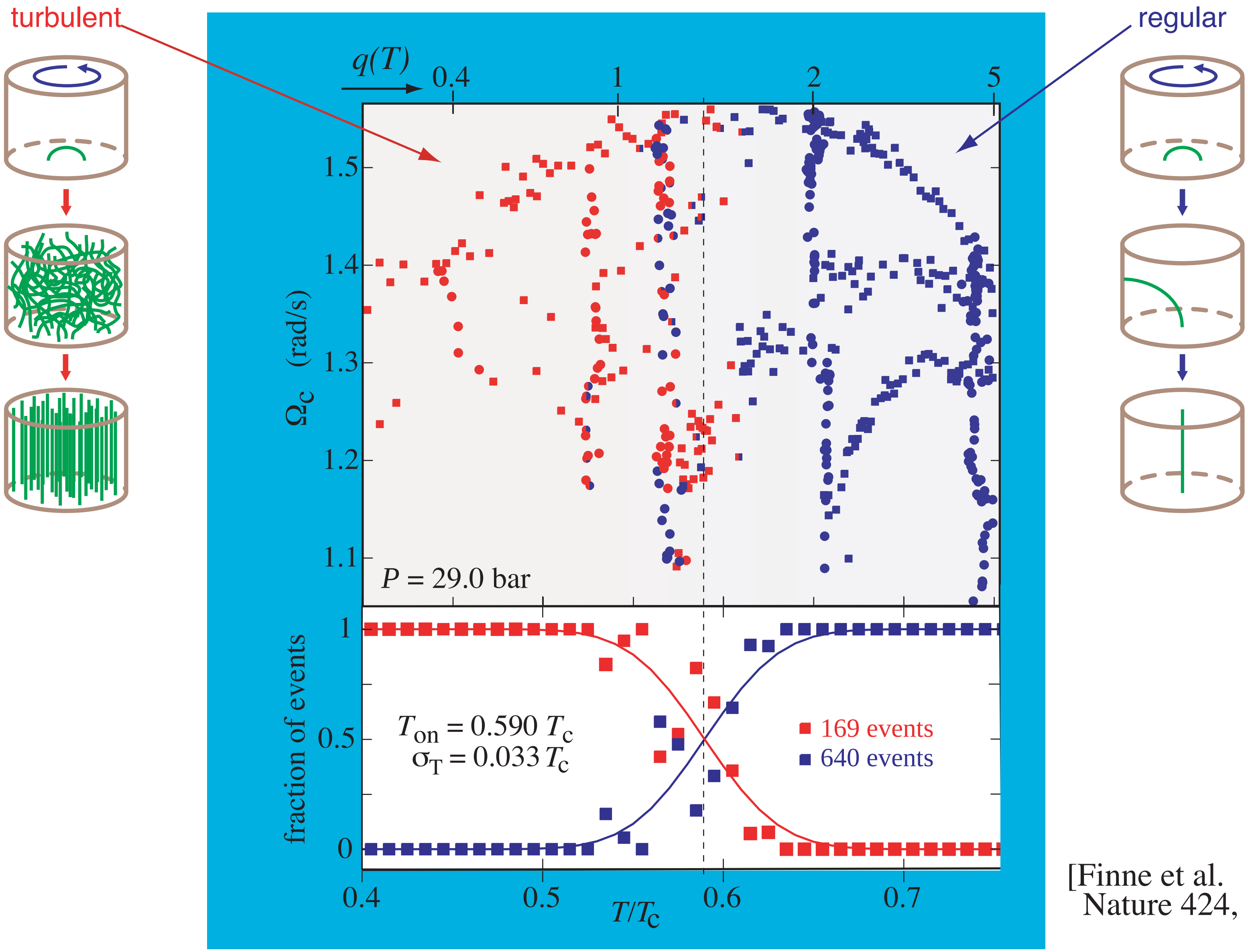
$\mathrm{Re}_{\alpha}(T_c) = 0$ $\mathrm{Re}_{\alpha}(T \sim 0.6\, T_c) \sim 1$ $\mathrm{Re}_{\alpha}(0) = \infty$

laminar vortex flow turbulent vortex flow

turbulent and laminar vortex evolution



TURBULENT VORTEX GENERATION



[Finne et al.
Nature 424, 1022 (2003)]